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Properly even harmonious labelings of disconnected graphs

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Abstract

A graph G with q edges is said to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. If G is a tree, exactly one label may be used on two vertices. Over the years, many variations of harmonious labelings have been introduced.

We study a variant of harmonious labeling. A function f is said to be a properly even harmonious labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2(q-1)$ and the induced function f^* from the edges of G to $0, 2, \ldots, 2(q-1)$ defined by $f^*(xy) = f(x) + f(y) (\bmod 2q)$ is bijective. This paper focuses on the existence of properly even harmonious labelings of the disjoint union of cycles and stars, unions of cycles with paths, unions of squares of paths, and unions of paths.

Keywords: Properly even harmonious labelings; Even harmonious labelings; Harmonious labelings; Graph labelings

1. Introduction

A vertex *labeling* of a graph G is a mapping f from the vertices of G to a set of elements, often integers. Each edge xy has a label that depends on the vertices x and y and their labels $f(x)$ and $f(y)$. Graph labeling methods began with Rosa [1] in 1967. In 1980, Graham and Sloane [2] introduced harmonious labelings in connection with error-correcting codes and channel assignment problems. There have been three published papers on even harmonious graph labelings by Sarasija and Binthiya [3,4] and Gallian and Schoenhard [5]. In this paper we focus on the existence of properly even harmonious labelings for the disjoint union of cycles and stars, unions of cycles with paths, unions of squares of paths, and unions of paths.

An extensive survey of graph labeling methods is available online [6]. We follow the notation in [6].

2. Preliminaries

Definition 2.1. A graph G with q edges is said to be *harmonious* if there exists an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y)$

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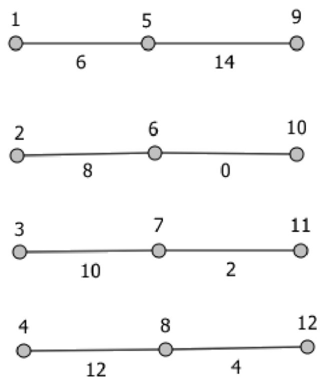

Fig. 1. $4P_3$, (mod 16), Theorem 3.1.

(mod q), the resulting edge labels are distinct. When G is a tree, exactly one edge label may be used on two vertices.

Definition 2.2. A function f is said to be an *even harmonious* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2q$ and the induced function f^* from the edges of G to $0, 2, \ldots, 2(q-1)$ defined by $f^*(xy) = f(x) + f(y) (\text{mod } 2q)$ is bijective.

Because 0 and $2q$ are equal modulo $2q$, Gallian and Schoenhard [5] introduced the following more desirable form of even harmonious labelings.

Definition 2.3. An even harmonious labeling of a graph G with q edges is said to be a *properly even harmonious labeling* if the vertex labels belong to $\{0, 2, \ldots, 2q-2\}$.

Definition 2.4. A graph that has a (properly) even harmonious labeling is called *(properly) even harmonious* graph.

Bass [7] has observed that for connected graphs, a harmonious labeling of a graph with q edges yields an even harmonious labeling by multiplying each vertex label by 2 and adding the vertex labels modulo $2q$. Gallian and Schoenhard [5] showed that for any connected even harmonious labeling, we may assume the vertex labels are even. Therefore we can obtain a harmonious labeling from a properly even harmonious labeling by dividing each vertex label by 2 and adding the vertex labels modulo q. Consequently, we focus our attention on disconnected graphs.

3. Disconnected graphs

Theorem 3.1. *The graph nP_m is properly even harmonious if n is even and $m \geq 2$.*

Proof. The modulus is $2(mn - n)$.

Drawing the graph as shown in Fig. 1, label the vertices starting with the top left corner to the bottom left corner with $1, 2, \ldots, n$ then label the second vertex of the first path $n + 1$, continuing to label the second vertices of all n paths consecutively with $n + 2, n + 3, \ldots, 2n$. The third vertex of the first path will be labeled $2n + 1$, and the remaining vertices are labeled consecutively with $2n + 2, 2n + 3, \ldots, 3n$. The mth vertices of the n paths are labeled with $mn - n + 1, mn - n + 2, \ldots, mn$.

Reading the edge labels vertically from top to bottom and left to right, we see that they begin with $n+2$. Increasing by 2 each time, they end with $(2mn - n) \bmod (2mn - 2n) = n$. So there is no overlap of edge labels.

To see that there is no duplicated vertex labels, notice that the vertex labels are $1, 2, \ldots, m$ and $mn \leq 2mn - 2n$, which simplifies to $0 \leq mn - 2n$. This is clearly true since $m \geq 2$. □

Gallian and Schoenhard [5] proved that the graph $P_n \cup K_{m,2}$ is properly even harmonious for n odd and $1 < n < 4m + 3$. This next theorem extends these results.

Theorem 3.2. *The graph $P_n \cup K_{s,t}$ is properly even harmonious for $n > 1$.*

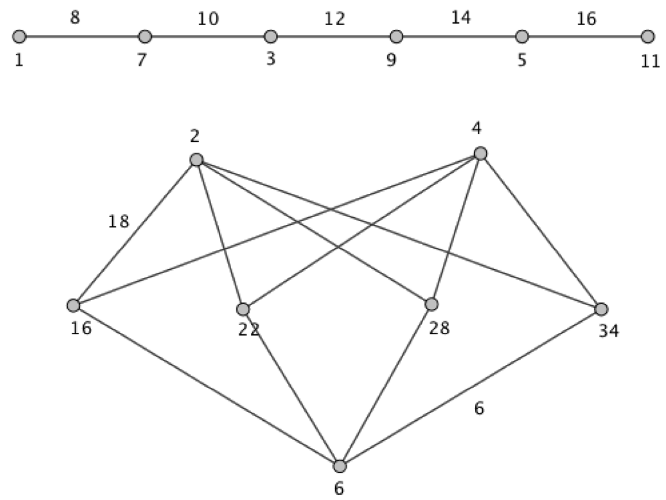

Fig. 2. $P_6 \cup K_{4,3}$, (mod 34), Theorem 3.2.

Proof. The modulus is $2n + 2st - 2$. We may assume that $s \geq t$.

- Case 1: n is even
  Starting with the first vertex of P_n use $1, 3, 5, \ldots, n$ skipping a vertex at each step and wrapping around.
  This gives the edge labels for P_n : $n + 2, n + 4, \ldots, 3n - 2 \bmod (2n + 2st - 2)$.
  Since these are increasing before reaching the modulus and less than $n + 2$ after they exceed the modulus there is no duplication of edge labels.
  Label the vertices in the partite set with t vertices with $2, 4, \ldots, 2t$.
  Label the vertices in the partite set with s vertices with $3n - 2, 3n - 2 + 2t, \ldots, 3n - 2 + 2(s - 1)t$.
  This gives the edge labels for $K_{s,t}$: $3n, 3n + 2, \ldots, 3n - 2 + 2(s - 1)t + 2t = n \bmod (2n + 2st - 2)$, as desired. See Fig. 2.
  There is no overlap in the labels for $K_{s,t}$ when $3n - 2 > 2t$. When $2t \geq 3n - 2$ we adjust the vertex labels for $K_{s,t}$ by subtracting x from each vertex label of the partite set with t vertices and adding x to each vertex label in the partite set with s vertices where x is the smallest even integer such that $2x > 2t - (3n - 2)$. That avoids duplication of vertex labels and does not change the edge labels.
- Case 2: n is odd
  Starting with the first vertex of P_n use $1, 3, 5, \ldots, n$ skipping a vertex at each step and wrapping around.
  This gives the edge labels for P_n : $n + 3, n + 4, \ldots, 3n - 1 \bmod (2n + 2st - 2)$.
  Label the vertices in the partite set with t vertices with $2, 4, \ldots, 2t$.
  Label the vertices in the partite set with s vertices with $3n - 1, 3n - 1 + 2t, \ldots, 3n - 1 + 2(s - 1)t$.
  This gives the edge labels for $K_{s,t}$: $3n + 1, 3n + 3, \ldots, 3n - 1 + 2(s - 1)t + 2t + 2 = n + 1 \bmod (2n + 2st - 2)$, as desired.
  There is no overlap in the labels for $K_{s,t}$ when $3n - 1 > 2t$. When $2t \geq 3n - 1$ we adjust the vertex labels for $K_{s,t}$ by subtracting x from each vertex label of the partite set with t vertices and adding x to each vertex label in the partite set with s vertices where x is the smallest even integer such that $2x > 2t - (3n - 1)$. That avoids duplication of vertex labels and does not change the edge labels. □

The method used in Theorem 3.2 can easily adapted to prove the following theorem.

Theorem 3.3. *The graph $C_n \cup K_{s,t}$ is properly even harmonious for odd $n > 1$.*

We next look at unions involving paths and stars.

Theorem 3.4. *The graph $P_n \cup S_{t_1} \cup S_{t_2} \cup \cdots \cup S_{t_k}$ is properly even harmonious for $n > 2$ and at least one t_i is greater than 1.*

Proof. The modulus is $2n + 2t_1 + 2t_2 + \cdots + 2t_k - 2$. We may assume that $t_1 \leq t_2 \leq \cdots \leq t_k$.

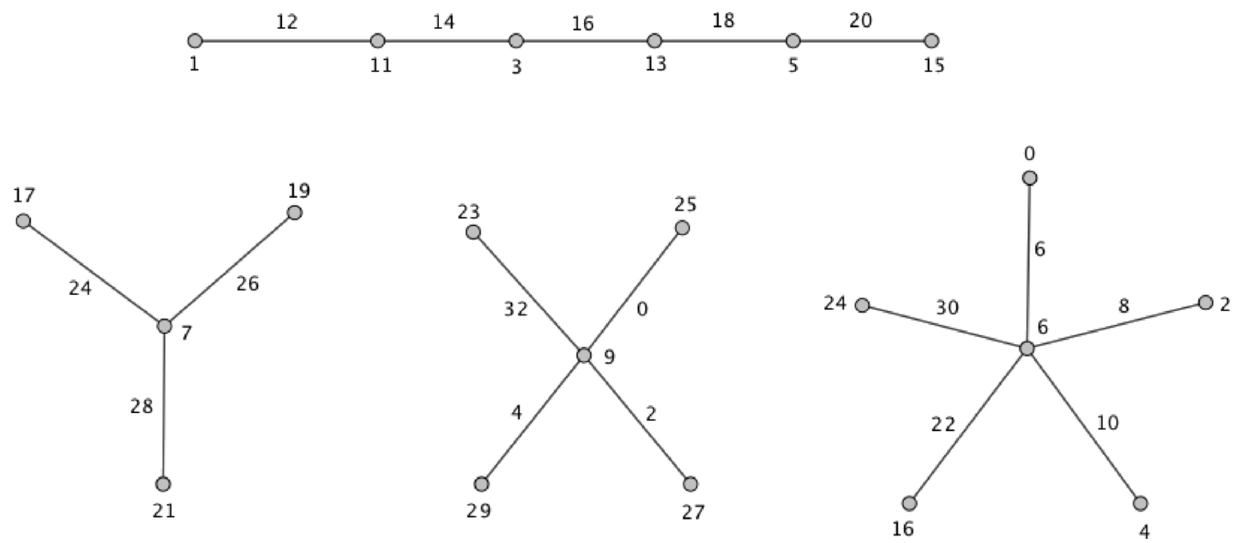

Fig. 3. $P_6 \cup S_5 \cup S_4 \cup S_3$, (mod 34), Theorem 3.4.

- Case 1: n is even

  The case that $k = 1$ was done in [5].

  Step 1: Label the first $n/2$ vertices of P_n using $1, 3, 5, \ldots, n - 1$ skipping a vertex at each step.

  Step 2: Label the remaining $n/2$ vertices of P_n starting at the second vertex using $n - 1 + 2k, n - 1 + 2k + 2, \ldots, 2n + 2k - 3$ skipping a vertex at each step.

  Step 3: Label the centers of the first $k - 1$ stars with: $n + 1, n + 3, \ldots, n + 1 + 2(k - 2)$.

  Step 4: Label the endpoints of S_{t_1} with $2n + 2k - 1, 2n + 2k + 1, \ldots, 2n + 2k - 1 + 2(t_1 - 1)$.

  Step 5: For $S_{t_2}, S_{t_3}, \ldots, S_{t_{k-1}}$ label the successive endpoints of each star by incrementing the last endpoint label previously used by 2. See Fig. 3.

  This gives distinct vertex and edge labels for $P_n \cup S_{t_1} \cup S_{t_2} \cup \cdots \cup S_{t_{k-1}}$.

  Step 6: Label the center vertex of S_{t_k} with $(n + 2k)/2$ if it is even and otherwise add 1. For each edge label x not yet used label an endpoints of S_{t_k} with the difference of the center vertex label and x.

  The choice of the label for center vertex of S_{t_k} ensures that there are no duplicate vertex or edge labels for $P_n \cup S_{t_1} \cup S_{t_2} \cup \cdots \cup S_{t_k}$.
- Case 1: n is odd

  This case is conceptually identical to Case 1. □

The method used in Theorem 3.4 can easily adapted to prove the following theorem. See Fig. 4.

Theorem 3.5. *The graph* $C_n \cup S_{t_1} \cup S_{t_2} \cup \cdots \cup S_{t_k}$ *is properly even harmonious for odd* $n > 2$ *and at least one* t_i *is greater than* 1.

Recall, a *caterpillar* is a graph obtained by starting with a path and adding one or more pendant edges to the vertices of the path. In the following theorems, we draw caterpillars as bipartite sets in a zigzag vertical fashion with one partite set on the left and the other partite set on the right. We denote these sets in the ith caterpillar by L_i and R_i and their sizes by l_1 and r_i. Without loss of generality, we may assume $l_i \le r_i$. We denote a caterpillar of path length m with t pendant edges, l vertices in the left partite set, and r vertices in the right partite set by $Cat_m{}^{+t}(l, r)$.

Theorem 3.6. *The graph consisting of* 2 *caterpillars is properly even harmonious.*

Proof. Let m denote the number of edges of a graph consisting of 2 caterpillars. The modulus is $2m$.

Step 1: With the vertices arranged into bipartite sets as described above, label L_1 beginning at the top with $0, 2, \ldots, 2l_1 - 2$. Then start at the top of R_1 and continue with $2l_1, 2l_1 + 2, \ldots, 2l_1 + 2r_1 - 2$. The corresponding edges are $2l_1, 2l_1 + 2, \ldots, 4l_1 + 2r_1 - 4$ as shown in Fig. 5.

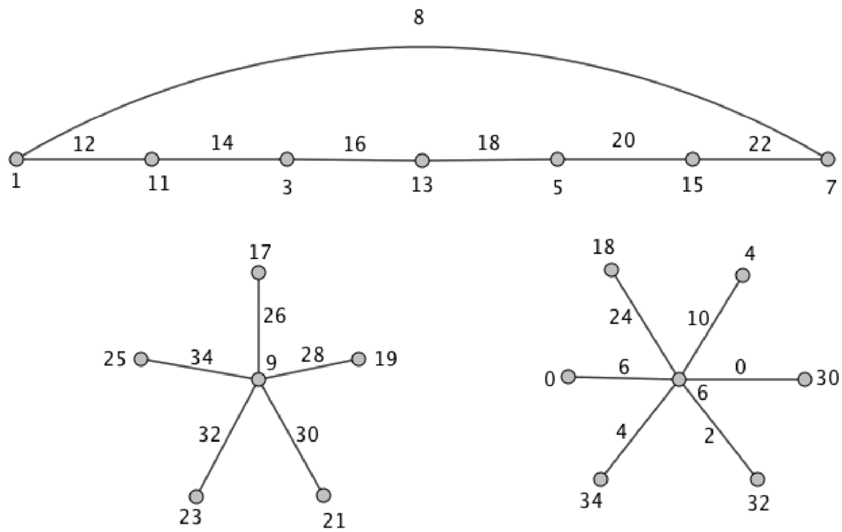

Fig. 4. $C_7 \cup S_5 \cup S_6$, (mod 36), Theorem 3.5.

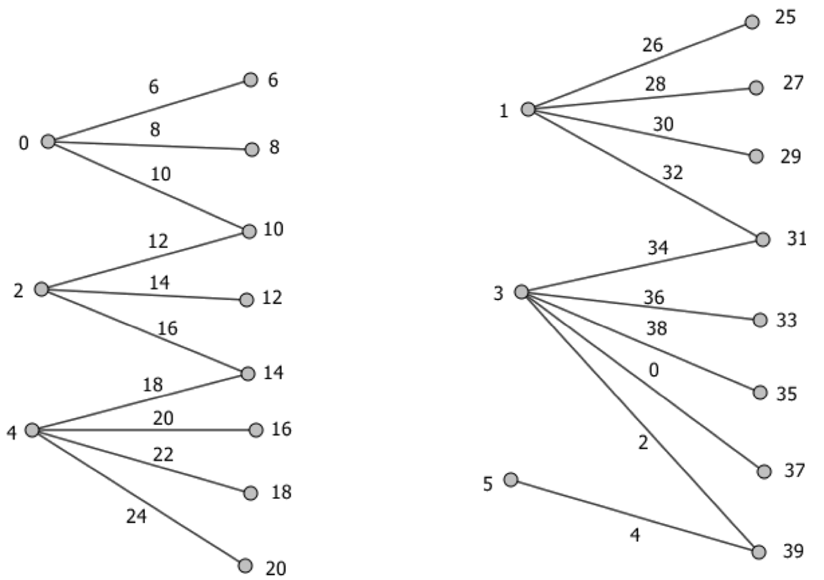

Fig. 5. $Cat_6{}^{+6}(3, 8) \cup Cat_5{}^{+5}(3, 8)$, (mod 40), Theorem 3.6.

Step 2: Label the vertices of second caterpillar beginning at the top of L_2 with $1, 3, \ldots, 2l_2 - 1$. Starting at the top label R_2 with $4l_1 + 2r_1 - 3, 4l_1 + 2r_1 - 1, \ldots, 4l_1 + 2r_1 + 2r_2 - 5$. The corresponding edge labels are $4l_1 + 2r_1 - 2, 4l_1 + 2r_1, \ldots, 4l_1 + 2r_1 + 2l_2 + 2r_2 - 6$.

If there are k overlaps in vertex labels, shift the labels by adding $2k$ to the labels on R_2 and subtracting $2k$ from L_2 as shown in Fig. 6. Choose the set based on which will give a properly even harmonious labeling. The edge labels are unchanged.

In the first caterpillar, the edge labels are distinct. This follows from the observation that they form an arithmetic progression with common difference 2 and the largest gap between two edge labels is less than the modulus. □

Theorem 3.7. *The graph consisting of* 3 *caterpillars is properly even harmonious.*

Proof. Let q denote the number of edges of graph consisting of 3 caterpillars. The modulus is $2q$.

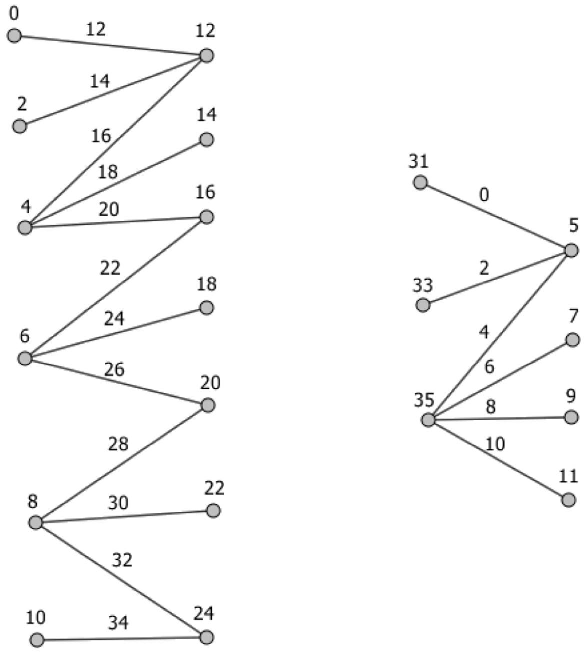

Fig. 6. $Cat_9{}^{+4}(6, 7) \cup Cat_4{}^{+3}(3, 4)$, (mod 36), $k = 3$, Theorem 3.6.

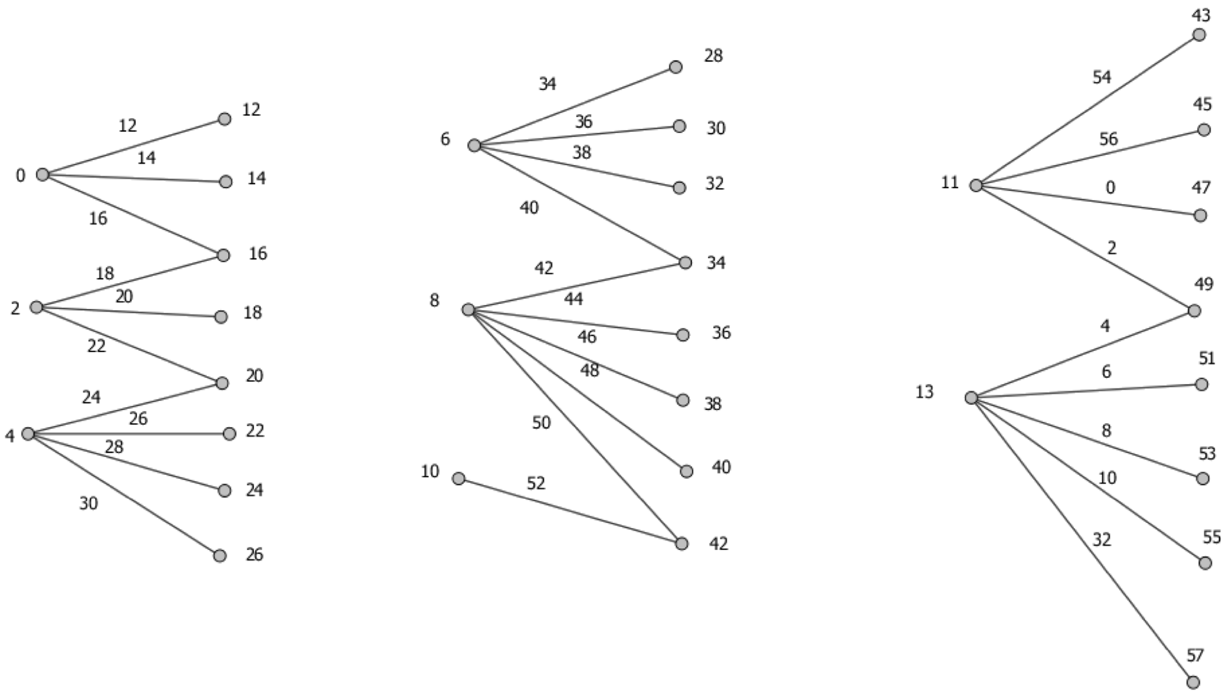

Fig. 7. $Cat_6{}^{+4}(3, 8) \cup Cat_5{}^{+5}(3, 8) \cup Cat_4{}^{+5}(2, 8)$, (mod 58), Theorem 3.7.

Arrange into 3 bipartite sets as described in the proof of Theorem 3.7 and denote them as L_i and R_i where $|L_i| = l_i$ and $|R_i| = r_i$ for $i = 1, 2, 3$. Without loss of generality, we may assume $l_i \leq r_i$.

For the third caterpillar, arrange the graph such that the first pendant vertex is contained in the right bipartite set.

Label L_1 as $0, 2, \ldots, 2l_1 - 2$. Then label L_2 as $2l_1, 2l_1 + 2, \ldots, 2l_1 + 2l_2 - 2$. Now go back to R_1 and label as $2l_1 + 2l_2, 2l_1 + 2l_2 + 2, \ldots, 2l_1 + 2l_2 + 2r_1 - 2$. Continue to R_2 and label as $2l_1 + 2l_2 + 2r_1, 2l_1 + 2l_2 + 2r_1 + 2, \ldots, 2l_1$

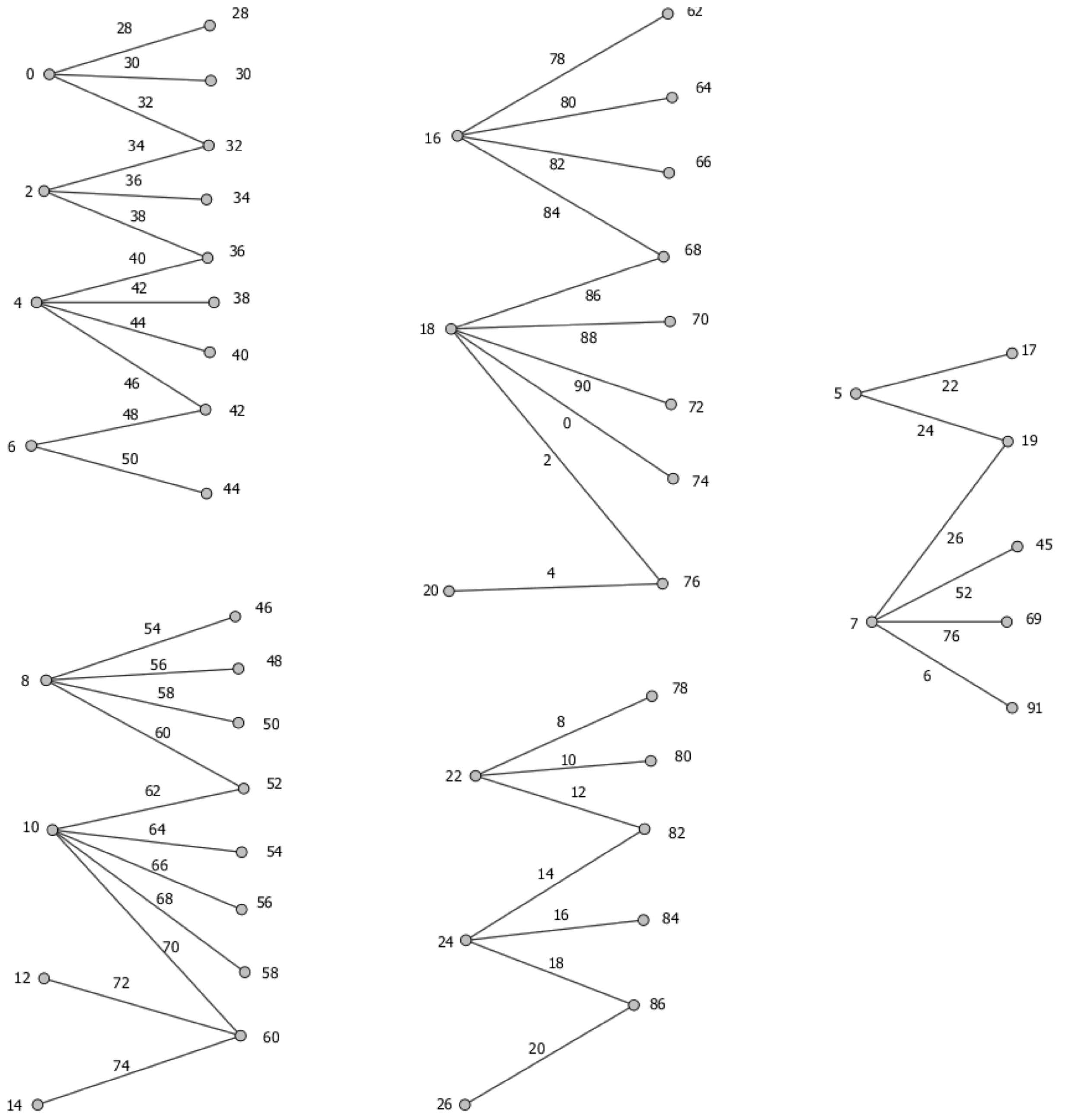

Fig. 8. $Cat_8{}^{+4}(4, 9) \cup Cat_5{}^{+6}(4, 8) \cup Cat_5{}^{+5}(3, 8) \cup Cat_5{}^{+2}(3, 5) \cup Cat_4{}^{+2}(2, 5)$, (mod 92), Theorem 3.8.

$+2l_2 + 2r_1 + 2r_2 - 2$. The corresponding edge labels on the first caterpillar are $2l_1 + 2l_2, 2l_1 + 2l_2 + 2, \ldots, 4l_1 + 2l_2 + 2r_1 - 4$. The edge labels on the second caterpillar are $4l_1 + 2l_2 + 2r_1, 4l_1 + 2l_2 + 2r_1 + 2, \ldots, 4l_1 + 4l_2 + 2r_1 + 2r_2 - 4$.

Notice that the edge label $4l_1 + 2l_2 + 2r_1 - 2$ skipped as shown in Fig. 7 is 32.

Label the third caterpillar beginning on L_3 as $1, 3, \ldots, 2l_3 - 1$. Label R_3 as $4l_1 + 4l_2 + 2r_1 + 2r_2 - 3, 4l_1 + 4l_2 + 2r_1 + 2r_2 - 1, \ldots, 4l_1 + 4l_2 + 2r_1 + 2r_2 + 2r_3 - 9$ skipping the first pendant vertex. Label that pendant vertex as $4l_1 + 2l_2 + 2r_1 - 3$. The corresponding edge label will now be $4l_1 + 2l_2 + 2r_1 - 2$ as required.

If there are k overlaps in vertex labels, shift the labels by adding $2k$ to the labels on one bipartite set and subtracting $2k$ from the other bipartite set. Choose the set based on which will give a properly even harmonious labeling. In the

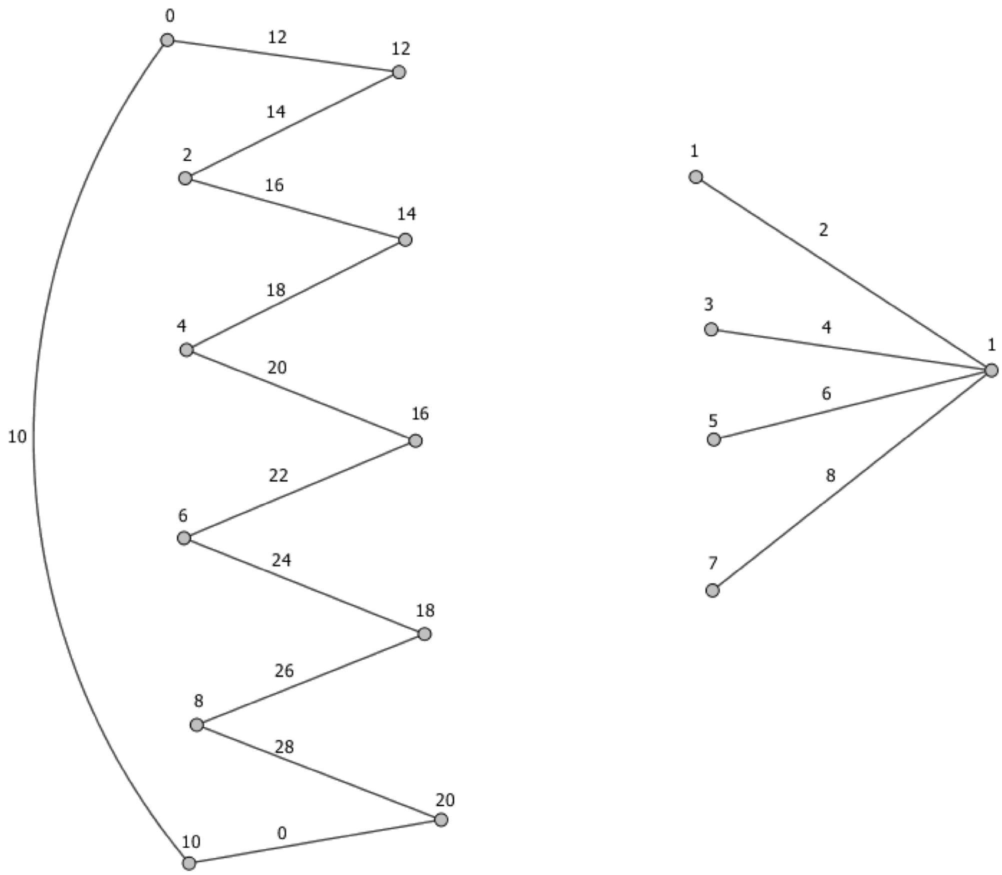

Fig. 9. $C_{11} \cup Cat_3{}^{+2}(4, 1)$, (mod 30), vertex label duplicated with $r = 1$, Theorem 3.9.

case that this overlap occurs with the first pendant vertex, it may be necessary to repeat this process. The edges are labeled the same as previous labeling.

Clearly, there is no overlap in vertex labels for the first two caterpillars. For the edge labels, notice that they form an arithmetic progression with common difference of 2 and the largest gap between two edge labels before we apply the modulus is less than the modulus. □

Similarly, we outline an algorithm for a properly even harmonious labeling of n caterpillars. Due to the fact that $n-2$ edge labels will be skipped in the process of labeling the first $n-1$ caterpillars, we must have one of the caterpillars with $n-2$ vertices of degree one. We will label this particular caterpillar last in order to pick up the skipped edge labels. In the following theorem, arrange the n caterpillars into bipartite sets. Denote these sets as L_i and R_i where $|L_i| = l_i$ and $|R_i| = r_i$ for $i = 1, 2, \ldots, n$. Without loss of generality, we may assume $l_i \le r_i$.

Theorem 3.8. *The graph consisting of n caterpillars is properly even harmonious if one caterpillar has at least $n-2$ vertices of degree* 1.

Proof. Let q denote the number of edges of the graph consisting of n caterpillars. The modulus is $2q$.

Label the first $n-1$ components as in Theorem 3.7 for the even vertex labels. Let the nth caterpillar be the one with at least $n-2$ vertices of degree 1. For the nth caterpillar, use odd vertex labels, shifting when necessary. Leave $n-2$ vertices of degree 1 unlabeled in order to pick up the skipped edge labels. Label these as needed. □

Fig. 8 shows this algorithm for 5 caterpillars and is easily extended to n caterpillars.

We now look at other families of disconnected graphs involving caterpillars. The first of these is an odd cycle with a caterpillar.

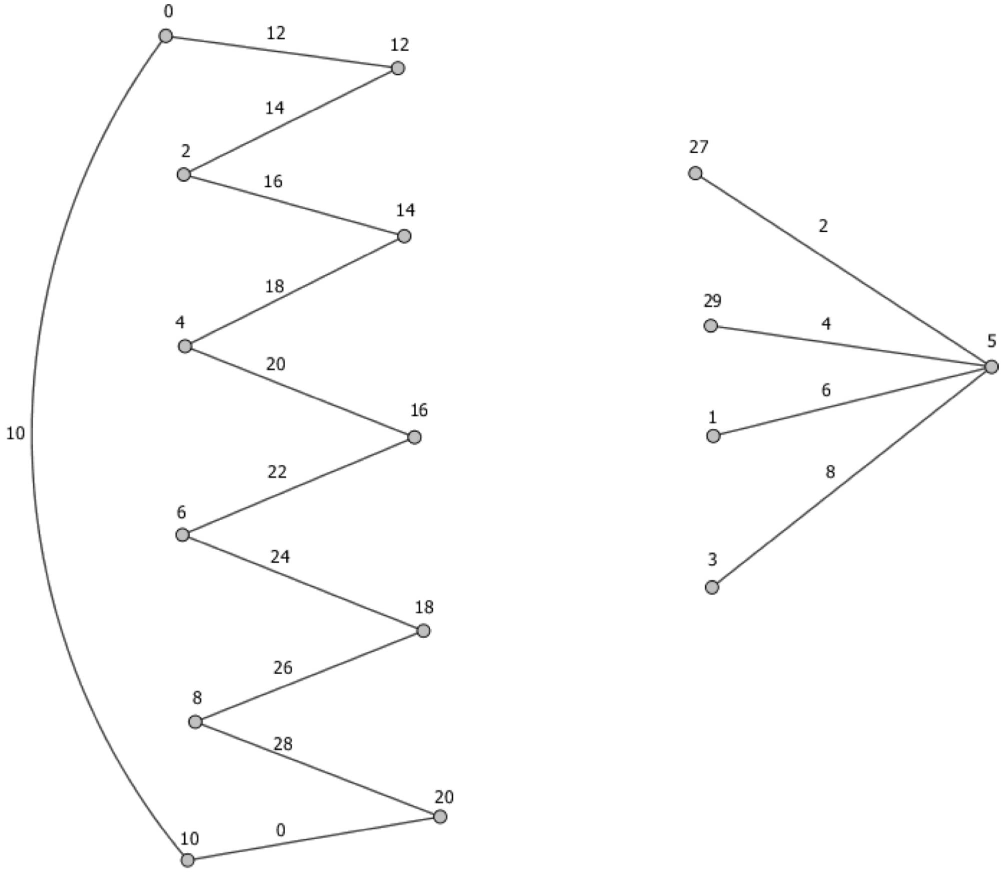

Fig. 10. $C_{11} \cup Cat_3{}^{+2}(4, 1)$, (mod 30), after shifting by $4r$, Theorem 3.9.

Definition 3.1. We call a graph G *pseudo-bipartite* if G is not bipartite but the removal of one edge results in a bipartite graph.

An example of this is found in Fig. 9. We use a pseudo-bipartite arrangement for the odd cycle and a bipartite arrangement for the caterpillar.

Theorem 3.9. *$C_m \cup Cat_n{}^{+k}(l, r)$ is properly even harmonious for m odd, $m > 2$, $n > 1$.*

Proof. The modulus is $2m+2n+2k-2$. Arrange C_m into a pseudo-bipartite set as shown in Fig. 9 and the caterpillar into a bipartite set. Let L_1 be the pseudo-bipartite set of C_m on the left and R_1 be the set on the right. Likewise, let L_2 be the bipartite set of $Cat_n{}^{+k}(l, r)$ on the left and R_2 be the set on the right where $|L_i| = l_i$ and $|R_i| = r_i$ for $i = 1, 2$.

Step 1: Begin labeling the vertices of L_1 with $0, 2, \ldots, 2l_1 - 2$ and then label the vertices of R_1 with $2l_1, 2l_1 + 2, \ldots, 2l_1 + 2r_1 - 2$. The corresponding edge labels are $2l_1 - 2, 2l_1, \ldots, 4l_1 + 2r_1 - 4$.

Step 2: Label the vertices of L_2 with $1, 3, \ldots, 2l_2 - 1$ and then label the vertices of R_2 with $4l_1 + 2r_1 - 3, 4l_1 + 2r_1 - 1, \ldots, 4l_1 + 2r_1 + 2r_2 - 5$.

If this labeling causes k duplications as shown in Fig. 9, subtract $4k$ from all vertex labels on L_2 and add $4k$ to all vertex labels on R_2 as in Fig. 10. The edges will remain the same. □

Although the algorithm used in Theorem 3.9 works only for odd cycles, there is a special case of an even cycle, C_4, for which a properly even harmonious labeling can be found with a caterpillar.

Theorem 3.10. *$C_4 \cup Cat_m{}^{+n}(l, r)$ is properly even harmonious.*

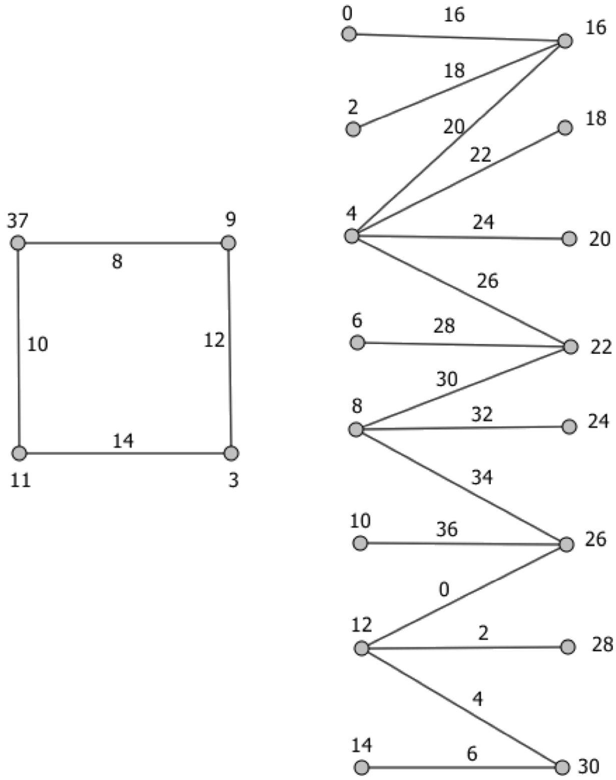

Fig. 11. $C_4 \cup Cat_9{}^{+7}(8, 8)$, (mod 38), Theorem 3.10.

Proof. The modulus is $2m + 2n + 6$. Arrange the caterpillar into bipartite sets L and R for left set and right set respectively. Let $|L| = l$ and $|R| = r$ and $l \leq r$. Label the vertices of L as $0, 2, \ldots, 2l - 2$ and label the vertices of R as $2l, 2l + 2, \ldots, 2l + 2r - 2$. The corresponding edge labels are $2l, 2l + 2, \ldots, 4l + 2r - 4$.

Label the vertices of C_4 consecutively as $2m + 2n + 5, 4l + 2r - 1, 3, 4l + 2r + 1$. The corresponding edge labels are $4l + 2r - 2, 4l + 2r + 2, 4l + 2r + 4, 4l + 2r$ as shown in Fig. 11.

To show there are no duplicate vertex labels in the caterpillar component, notice that the vertex labels are increasing with common difference 2 and the largest gap between two vertex labels is less than the modulus.

To verify there are no duplicate vertex labels in the C_4 component, we have $2m + 2n + 5 < 2m + 2n + 9$ and $4l + 2r - 1 < 4l + 2r + 1$. Since $4l + 2r = 4$ implies that $l = 1$ and $r = 0$, and likewise $4l + 2r = 2$ implies that $l = 0$ and $r = 1$, we know there is no duplication of labels on the C_4 component. □

Recall in Theorem 3.1 we looked at nP_m where n was even. We now look at $2P_n \cup 2P_m$. This consists of two copies of P_n and two copies of P_m. Clearly, the cases are covered by Theorem 3.1 when $m = n$, so we focus on $m < n$.

Theorem 3.11. *$2P_m \cup 2P_n$ is properly even harmonious for $m, n > 1$.*

Proof. The modulus is $4m + 4n - 8$. By Theorem 3.1, we may assume that $m < n$.

Step 1: Arrange the paths as shown in Figs. 12 and 13 where the first two paths correspond to $2P_m$ and the second two paths correspond to $2P_n$.

Step 2: Label the first path of length m as $0, 2, \ldots, 2m - 4$ leaving the last vertex unlabeled. Label the second path of length m with $1, 3, \ldots, 2m - 3$ leaving the last vertex unlabeled.

Step 3: Label the vertices of the first path with length n as $2m - 2, 2m, \ldots, 2m + 2n - 6$ leaving the last vertex unlabeled. Label the vertices of the second path of length n with $2m - 1, 2m + 1, \ldots, 2m + 2n - 5$ leaving the last vertex unlabeled.

Notice that the edge labels are all the even integers between 0 and $4m + 4n - 10$ except for the values $4m - 6, 4m - 4, 4m + 4n - 10, 4m + 4n - 8$.

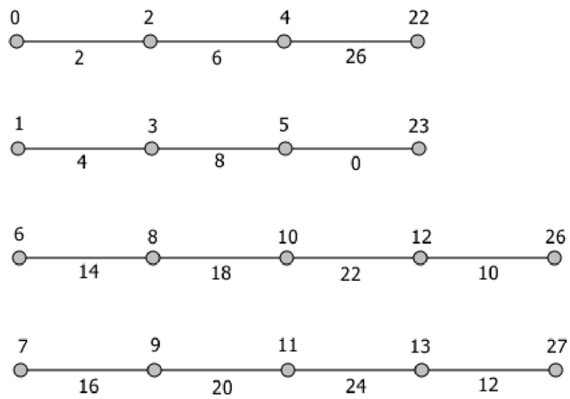

Fig. 12. $2P_4 \cup 2P_5$, (mod 28), Theorem 3.11.

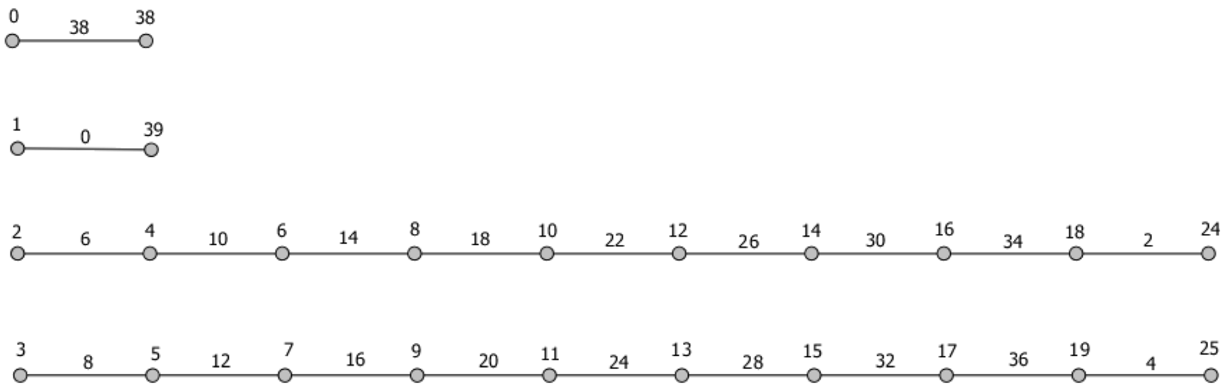

Fig. 13. $2P_2 \cup 2P_{10}$, (mod 40), Theorem 3.11.

Label the remaining four vertices as follows. On the first path of length m, use $2m + 4n - 6$. On the second path of length m use $2m + 4n - 5$. This picks up the missing edge labels of $4m + 4n - 10$ and $4m + 4n - 8$.

Label the last vertex of the first path of length n with $2m - 2n = 6m + 2n - 8$ and the last vertex of the second path of length n with $2m - 2n + 1 = 6m + 2n - 7$. This will pick up the remaining missing edge labels of $4m - 6$ and $4m - 4$. See Figs. 12 and 13.

Since the vertex labels, excluding the right end points of the four paths, are strictly increasing by increments of two and less than the modulus they are distinct. Moreover, the four right end vertex labels are distinct and less than the modulus so there is no wrap around. □

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