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Even harmonious labelings of disjoint graphs with a small component

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Abstract

A graph G with q edges is said to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. If G is a tree, exactly one label may be used on two vertices. Over the years, many variations of harmonious labelings have been introduced.

We study a variant of harmonious labeling. A function f is said to be a properly even harmonious labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2(q-1)$ and the induced function f^* from the edges of G to $0, 2, \ldots, 2(q-1)$ defined by $f^*(xy) = f(x) + f(y) \pmod{2q}$ is bijective. We investigate the existence of properly even harmonious labelings of families of disconnected graphs with one of C_3, C_4, K_4 or W_4 as a component.

Keywords: Properly even harmonious labelings; Even harmonious labelings; Harmonious labelings; Graph labelings

1. Introduction

A vertex *labeling* of a graph G is a mapping f from the vertices of G to a set of elements, often integers. Each edge xy has a label that depends on the vertices x and y and their labels $f(x)$ and $f(y)$. Graph labeling methods began with Rosa [1] in 1967. In 1980, Graham and Sloane [2] introduced harmonious labelings in connection with error-correcting codes and channel assignment problems. There have been three published papers on even harmonious graph labelings by Sarasija and Binthiya [3,4] and Gallian and Schoenhard [5]. In [6] we focus on the existence of properly even harmonious labelings for the disjoint union of cycles and stars, unions of cycles with paths, unions of squares of paths, and unions of paths. In this paper we investigate the existence of properly even harmonious labelings of families of disconnected graphs with one of C_3, C_4, K_4 or W_4 as a component.

An extensive survey of graph labeling methods is available online [7]. We follow the notation in [7].

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2. Preliminaries

Definition 2.1. A graph G with q edges is said to be *harmonious* if there exists an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. When G is a tree, exactly one edge label may be used on two vertices.

Definition 2.2. A function f is said to be an *even harmonious* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2q$ and the induced function f^* from the edges of G to $0, 2, \ldots, 2(q-1)$ defined by $f^*(xy) = f(x) + f(y) \pmod{2q}$ is bijective.

Because 0 and $2q$ are equal modulo $2q$, Gallian and Schoenhard [5] introduced the following more desirable form of even harmonious labelings.

Definition 2.3. An even harmonious labeling of a graph G with q edges is said to be a *properly even harmonious labeling* if the vertex labels belong to $\{0, 2, \ldots, 2q-2\}$.

Definition 2.4. A graph that has a (properly) even harmonious labeling is called *(properly) even harmonious* graph.

Bass [8] has observed that for connected graphs, a harmonious labeling of a graph with q edges yields an even harmonious labeling by multiplying each vertex label by 2 and adding the vertex labels modulo $2q$. Gallian and Schoenhard [5] showed that for any connected even harmonious labeling, we may assume the vertex labels are even. Therefore, for a connected graph we can obtain a harmonious labeling from a properly even harmonious labeling by dividing each vertex label by 2 and adding the vertex labels modulo q. Consequently, we focus our attention on disconnected graphs.

3. Disconnected graphs

Definition 3.1. We define an *odd hairy cycle* as an odd cycle with one or more pendant edges attached.

Definition 3.2. We call a graph G *pseudo-bipartite* if G is not bipartite but the removal of one edge results in a bipartite graph.

We will use ${C_m}^{+n}$ to denote an m-cycle with n pendant edges attached.

To describe our labeling of ${C_m}^{+n}$ for m odd, we draw the m-cycle in a zigzag fashion as shown in Fig. 1. The pendant edges incident to the cycle vertices are drawn so that the endpoints are on the side opposite the cycle vertices. Ignoring the edge that joins the first and last vertices of the odd cycle, we have a bipartite graph with one partite set on the left (L) and the other on the right (R). We call this a *pseudo-bipartite* graph (Definition 3.2). Denote the number of edges in L as l and the number of edges in R as r.

It is convenient to denote an odd hairy cycle by specifying the sizes l and r of pseudo-bipartite sets L and R as ${C_m}^{+n}(l, r)$.

Theorem 3.1. *$C_4 \cup {C_m}^{+n}(l, r)$ is properly even harmonious.*

Proof. The modulus is $2m + 2n + 8$.

Arrange the pseudo-bipartite sets as described above and shown in Fig. 1. Label L of ${C_m}^{+n}(l, r)$ as $0, 2, \ldots, 2l-2$. Label R continuing with $2l, 2l+2, \ldots, 2l+2r-2$. The corresponding edge labels are $2l-2, 2l, \ldots, 4l+2r-4$.

Label the vertices of C_4 consecutively as $2m+2n+7, 4l+2r-1, 3, 4l+2r+1$. The corresponding edge labels are $4l+2r-2, 4l+2r+2, 4l+2r+4, 4l+2r$. See Fig. 1.

To verify there are no duplicate vertex labels in the C_4 component, notice that $2m+2n+7 < 2m+2n+11$ and $4l+2r-1 < 4l+2r+1$. Since $4l+2r = 4$ implies that $l = 1$ and $r = 0$, and likewise $4l+2r = 2$ implies that $l = 0$ and $r = 1$, we know there is no duplication of labels on the C_4 component. For the ${C_m}^{+n}(l, r)$ component, notice that the vertex labels are all increasing with common difference of 2. The largest gap between vertex labels is less than the modulus so there is no wrap around. $\square$

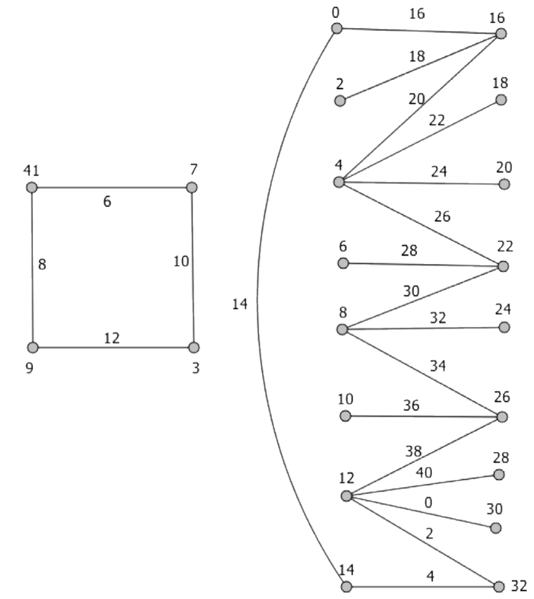

Fig. 1. $C_4 \cup C_9^{+8}(8, 9)$, (mod 42), Theorem 3.1.

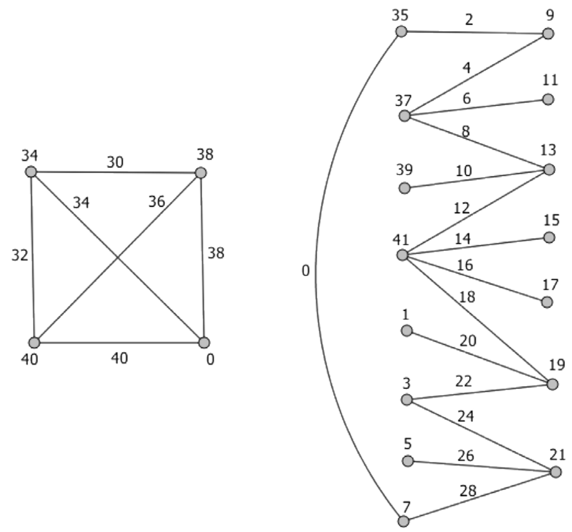

Fig. 2. $K_4 \cup C_9^{+6}(8, 7)$, (mod 42), $l \equiv 0 \pmod 4$, Theorem 3.2.

Using the same pseudo-bipartite arrangement for the odd hairy cycle as described previously (Definition 3.2), we can find a properly even harmonious labeling for the union of K_4 or $W_4 = C_4 + K_1$ with an odd hairy cycle.

Theorem 3.2. *$K_4 \cup C_m^{+n}(l, r)$ is properly even harmonious if $l \equiv 0, 2 \pmod 4$.*

Proof. The modulus is $2m + 2n + 12$.

Arrange $C_m^{+n}(l, r)$ into a pseudo-bipartite set as described for Theorem 3.1.

Step 1: Label the vertices of K_4 with $2m + 2n + 4, 2m + 2n + 8, 0, 2m + 2n + 10$ in this order as shown in Fig. 2. The edge labels are $2m + 2n, 2m + 2n + 2, \ldots, 2m + 2n + 10$.

Step 2: Label the vertices of L with $-l + 1, -l + 3, \ldots, l - 1$. Label the vertices of R with $l + 1, l + 3, \ldots, l + 2r - 1$. The corresponding edge labels are $0, 2, \ldots, 2l + 2r - 2 = 2m + 2n - 2$.

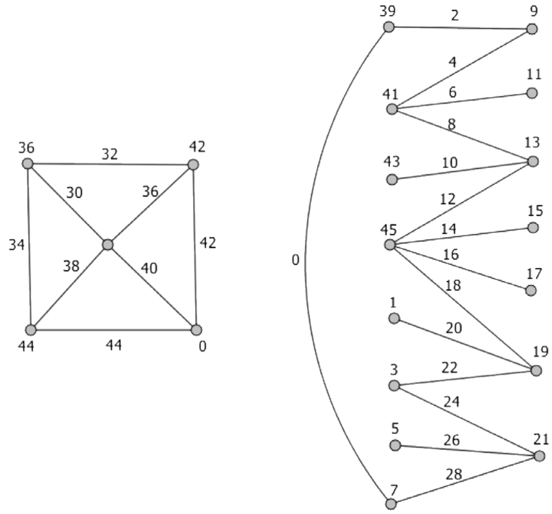

Fig. 3. $W_4 \cup C_9^{+6}(8, 7)$, (mod 46), $l \equiv 0 \pmod 4$, Theorem 3.3.

To verify that there is no duplication of vertex labels in the K_4 component, notice that $2m+2n+4 < 2m+2n+8 < 2m+2n+10 < 2m+2n+12$. For the C_m^{+n} component, the labels are increasing with common difference 2. The largest gap between vertex labels is less than the modulus so there is no wrap around. □

Theorem 3.3. *$W_4 \cup C_m^{+n}(l, r)$ is properly even harmonious for $m > 2$ odd if $l \equiv 0 \pmod 2$.*

Proof. The modulus is $2m + 2n + 16$. Arrange $C_m^{+n}(l, r)$ into a pseudo-bipartite set.

Step 1: Label the interior vertex of W_4 with $2m + 2n + 10$ and the perimeter vertices with $2m + 2n + 6, 2m + 2n + 12, 0, 2m + 2n + 14$ in this order. The corresponding edge labels are $2m + 2n, 2m + 2n + 2, \ldots, 2m + 2n + 14$.

Step 2: Label the vertices of L with $-l + 1, -l + 3, \ldots, l - 1$. Label the vertices of R with $l + 1, l + 3, \ldots, l + 2r - 1$ as shown in Fig. 3. The corresponding edge labels are $0, 2, \ldots, 2l + 2r - 2 = 2m + 2n - 2$.

To show there is no duplication of vertex labels in the W_4 component, notice that $2m + 2n + 6 < 2m + 2n + 10 < 2m + 2n + 12 < 2m + 2n + 14 < 2m + 2n + 16$. For the C_m^{+n} component, the labels are increasing with common difference 2. Since the largest gap between vertex labels is less than the modulus there is no wrap around. □

Theorem 3.4. *$C_4 \cup (P_n + \overline{K_2})$ is properly even harmonious for $n > 1$.*

Proof. The modulus is $6n + 6$. We do the cases $n = 2, 3, 4, 5$ ad hoc.

- Case 1: $n = 2$, the modulus is 18

  Step 1: Label the vertices of P_2 with 1, 3 and the vertices of degree n with 5, 9. The edge labels are $4, 6, \ldots, 12$.

  Step 2: Label the vertices of C_4 with 16, 0, 14, 4 in this order. The corresponding edge labels are 14, 16, 0, 2.
- Case 2: $n = 3$, the modulus is 24

  Step 1: Label the vertices of P_3 with 1, 5, 3 in this order. Label the vertices of degree n with 9, 15. The edge labels on P_3 are 6, 8 and the remaining edges are $10, 12, \ldots, 20$.

  Step 2: Label the vertices of C_4 with 20, 2, 0, 4 in order. The edge labels are 22, 2, 4, 0.
- Case 3: $n = 4$, the modulus is 30

  Step 1: Label the vertices of P_4 with 1, 5, 3, 7 and the vertices of degree n with 11, 19. The edge labels are $6, 8, \ldots, 26$.

  Step 2: Label the vertices of C_4 with 0, 2, 26, 4 in order. The corresponding edge labels are 28, 0, 2, 4.

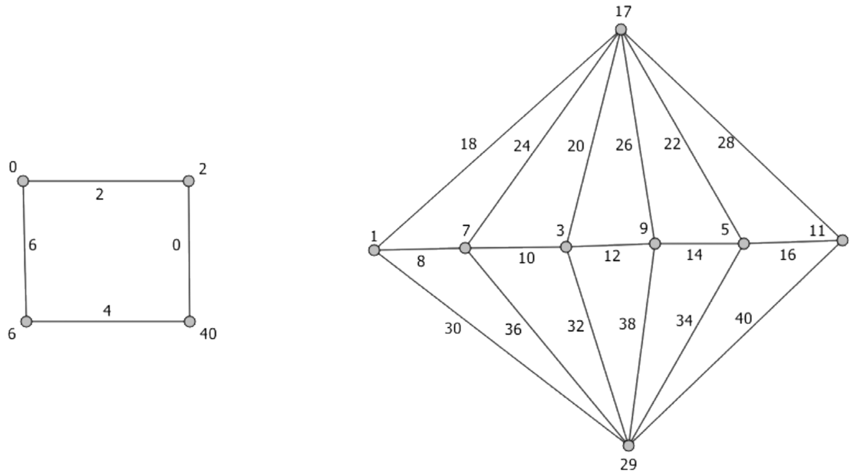

Fig. 4. $C_4 \cup (P_6 + \overline{K_2})$, (mod 42), Theorem 3.4.

- Case 4: $n = 5$, the modulus is 36

  Step 1: Label the vertices of P_5 with $1, 7, 3, 9, 5$ in this order. Label the vertices of degree n with 15, 25. The edge labels on P_5 are 8, 10, 12, 14 and the remaining edge labels are $16, 18, \dots, 34$.

  Step 2: Label the vertices of C_4 with 0, 2, 34, 6 in order. The corresponding edge labels are 0, 2, 4, 6.
- Case 5: n is even, $n \geq 6$

  Step 1: Label P_n starting with the first vertex with $1, 3, \dots, n-1$, skipping a vertex each time. Wrap around and label remaining vertices with $n+1, n+3, \dots, 2n-1$. Label the vertices of degree n with $3n-1$ and $5n-1$. The edge labels corresponding to the path are $n+2, n+4, \dots, 3n-2$; the remaining edge labels are $3n, 4n, 3n+2, 4n+2, \dots, 4n-2, 5n-2$ and $5n, 6n, 5n+2, 6n+2, \dots, 6n-2, 7n-2 = n-8$.

  Step 2: Label C_4 as $0, n-4, 6n+4, n$ in order. The corresponding edge labels are $n-4$, $7n = n-6$, $7n+4 = n-2$, n as shown in Fig. 4.

  To show there is no duplication in vertex labels, since $2n-1 < 3n-1 < 5n-1$, there is no duplication in the $(P_n + \overline{K_2})$ component. In C_4, we need $n-4 \neq 6n+4$. This simplifies to $n \neq 2$. Therefore there is no duplication in vertex labels.
- Case 6: n is odd, $n \geq 7$

  Step 1: Label P_n starting with the first vertex as $1, 3, \dots, n$ skipping a vertex each time. Wrap around and label remaining vertices as $n+2, n+4, \dots, 2n-1$. Label the vertices of degree n as $3n$ and $5n$. The edge labels corresponding to the path will be $n+3, n+5, \dots, 3n-1$ and the remaining edge labels are $3n+1, 4n+2, 3n+3, 4n+4, \dots, 5n-1, 4n$ and $5n+1, 6n+2, 5n+3, 6n+4, \dots, 7n-1, 6n$.

  Step 2: Label C_4 with $6n+4, n-3, 2, n-1$. The corresponding edge labels are $7n+1 = n-5$, $n-1$, $n+1$, $7n+3 = n-3$.

  To show there is no duplication in vertex labels in the $(P_n + \overline{K_2})$ component, notice that $2n-1 < 3n < 5n$. To verify that the vertex labels of C_4 are distinct for $n > 5$ observe that $2 < n-3 < n-1 < 6n+4 < 6n+6$. □

Theorem 3.5. *$K_4 \cup (P_n + \overline{K_m})$ is properly even harmonious if $n > 1$.*

Proof. The modulus is $2mn + 2n + 10$.

- Case 1: $n \equiv 0 \pmod 4$

  Start with the first vertex of P_n use $1, 3, 5, \dots, n$ skipping a vertex at each step and wrapping around.

  Label the m vertices of $\overline{K_m}$ with $3n-1, 5n-1, \dots, (2m+1)n-1$. This gives the edge labels for $P_n + \overline{K_m}$: $n+2, n+4, \dots, (2m+3)n-2 = 2mn+3n-2 \bmod (2mn+2n+10)$.

  Since these are increasing before reaching the modulus and less than $n+2$ after they exceed the modulus there is no duplication of edge labels.

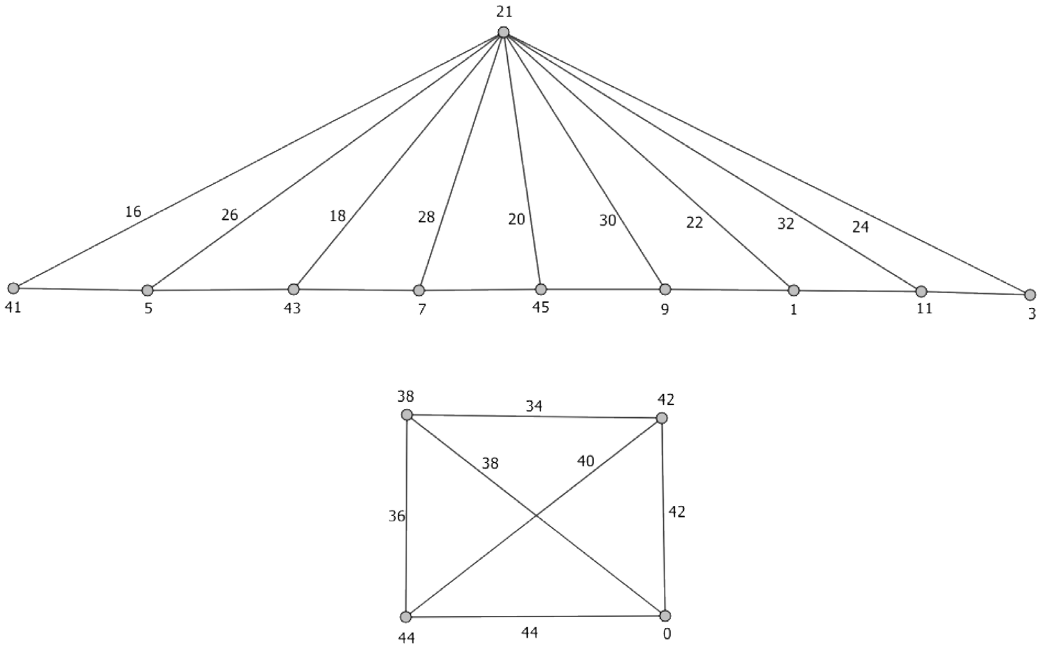

Fig. 5. $K_4 \cup (P_9 + \overline{K_1})$, (mod 46), $n \equiv 1 \pmod 4$, Theorem 3.5.

Label the vertices of K_4 with $x, x+4, x+8, x+6$ in order, where $x = (2mn+3n-4)/2$. This gives the edge labels $2x+4$, $2x+6$, $2x+8$, $2x+10$, $2x+12$, $2x+14 = 2mn+3n+10 \bmod (2mn+2n+10) = n$, as desired. Since $n \equiv 0 \pmod 4$ implies that x is even the vertex labels of K_4 cannot overlap with those used on $P_n + \overline{K_m}$.

- Case 2: $n \equiv 1 \pmod 4$

  Label the vertices of P_n as in Case 1.

  Label the m vertices of $\overline{K_m}$ with $3n, 5n, \ldots, (2m+1)n$. This gives the edge labels for $P_n + \overline{K_m}$: $n+3, \ldots, (2m+3)n - 1 = 2mn+3n-1 \bmod (2mn+2n+10)$.

  As in Case 1 there is no duplication of vertex or edge labels of $P_n + \overline{K_m}$. Label the vertices of K_4 with $x, x+4, x+8, x+6$ in order, where $x = (2mn+3n-3)/2 + mn + n + 5 = 2mn + 5n/2 + 7/2$. See Fig. 5.

  This gives the edge labels $2x+4, 2x+6, 2x+8, 2x+10, 2x+12, 2x+14 = 2mn+3n+11 \bmod (2mn+2n+10) = n+1$, as desired.

  Since these labels are even there is no overlap with the labels for $P_n + \overline{K_m}$.

- Case 3: $n \equiv 2 \pmod 4$

  This case is identical to Case 1 except that $x = (2mn+3n-4)/2 + mn + n + 5 = 2mn + 5n/2 + 3$. This results in even labels for K_4 but the same vertex labels as in Case 1 and shown in Fig. 6.

- Case 4: $n \equiv 3 \pmod 4$
  - Subcase i: $n = 3$

    The modulus is $6m + 16$.

    Label the vertices of P_3 with 5, 1, 3 in order. Label the vertices of $\overline{K_m}$ with $7, 13, \ldots, 6m+1$ in order. This gives the edge labels for $P_3 + \overline{K_m}$: $4, 6, \ldots, 6m+6$.

    Label the vertices of K_4 with $x, x+4, x+8, x+6$ in order, where $x = 6m+10$ when m is odd and $x = 3m+2$ when m is even. Modulo $6m+16$, this gives the edge labels $6m+8, \ldots, 6m+14, 0, 2$, as desired.
  - Subcase ii: $n > 3$

    Label the vertices of $P_n + \overline{K_m}$ as in Case 2.

    This gives the edge labels for $P_n + \overline{K_m}$: $n+3, \ldots, (2m+3)n - 1 = 2mn+3n-1 \bmod (2mn+2n+10)$. As in Case 1 there is no duplication of vertex or edge labels of $P_n + \overline{K_m}$.

    Label the vertices of K_4 with $x, x+4, x+8, x+6$ in order, where $x = (2mn+3n-3)/2$. This gives the edge labels $2x+4, 2x+6, 2x+8, 2x+10, 2x+12, 2x+14 = 2mn+3n+11 \bmod (2mn+2n+10) = n+1$, as desired.

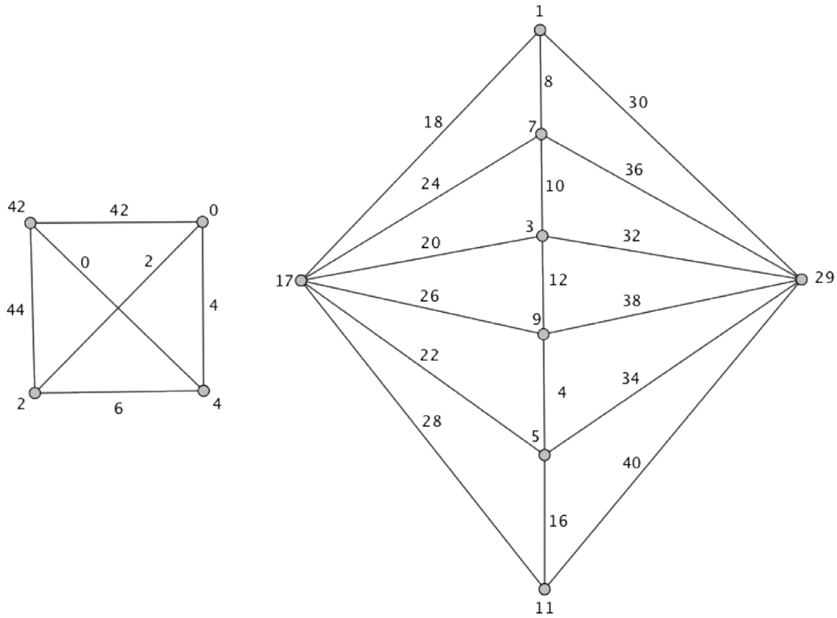

Fig. 6. $K_4 \cup (P_6 + \overline{K_2})$, (mod 46), $n \equiv 2 \pmod 4$, Theorem 3.5.

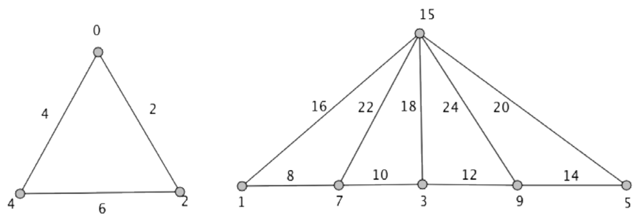

Fig. 7. $C_3 \cup (P_5 + \overline{K_1})$, (mod 24), $n \equiv 1 \pmod 4$, Theorem 3.7.

To prove that these labels do not overlap with the labels for $P_n + \overline{K_m}$ we need only check that none of these four have the form tn. To do so first observe that for $n > 3$, the inequality $x > (m+1)n$ holds, and for $n > 13$ the inequality $(m+2)n > x+8$ holds. Thus the only possible overlap of the labels for K_4 and $P_n + \overline{K_m}$ can occur only when $(m+2)n = (2mn+3n-3)/2 + k$ where $k = 0, 4, 6$, or 8 and $n = 7$ or 11. But this simplifies to $n = -3, 5, 9$, or 13, none of which are 3 mod 4. □

The labeling algorithm in Theorem 3.5 also yields the following two results.

Theorem 3.6. *$W_4 \cup (P_n + \overline{K_m})$ is properly even harmonious for $n > 1$.*

Theorem 3.7. *$C_3 \cup (P_n + \overline{K_m})$ and $C_5 \cup (P_n + \overline{K_m})$ are properly even harmonious for $n > 1$. (See Fig. 7.)*

Theorem 3.8. *$W_4 \cup P_n$ is properly even harmonious if $n > 1$.*

Proof. The modulus is $2n + 14$.

- Case 1: $n \equiv 0 \pmod 4$

  Step 1: Starting with the first vertex of P_n use $1, 3, 5, \ldots, n$ skipping a vertex at each step and wrapping around.

  This gives the edge labels for P_n : $n+2, n+4, \ldots, 3n-2 \bmod (2n+14)$. Since these are increasing before reaching the modulus and less than $n+2$ after they exceed the modulus there is no duplication of edge labels.

  Step 2: Label the rim vertices of W_4 with $x, x+6, x+10, x+8$ in order and the center of W_4 with $x+4$ where $x = 3n/2 - 2$.

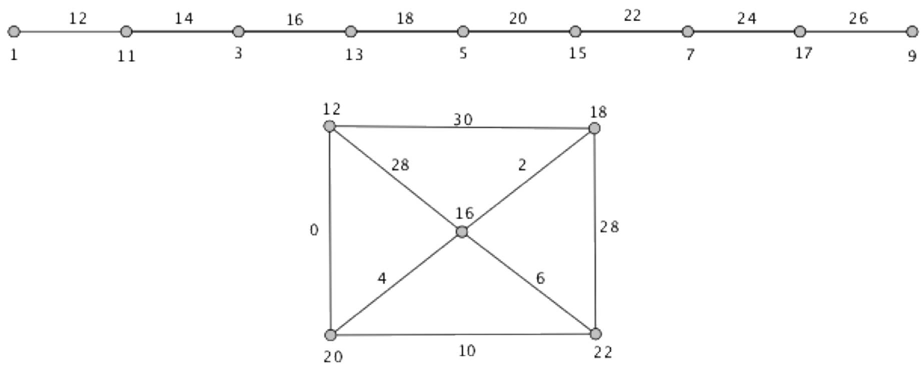

Fig. 8. $W_4 \cup P_9$, (mod 32), $n \equiv 1$ (mod 4), Theorem 3.8.

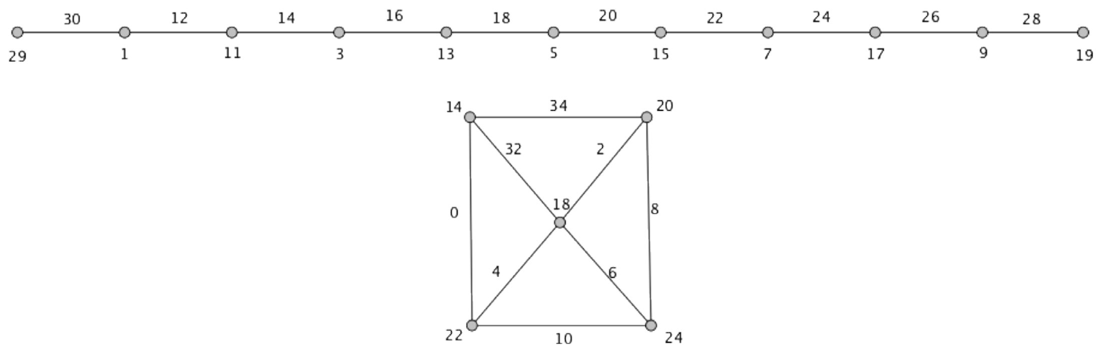

Fig. 9. $W_4 \cup P_{11}$, (mod 36), $n \equiv 3$ (mod 4), Theorem 3.8.

This gives the edge labels $2x+4$, $2x+6$, $2x+8$, $2x+10$, $2x+12$, $2x+14$, $2x+16$, $2x+18 = 3n+14$ mod $(2n+14) = n$, as desired.

Since $n \equiv 0$ mod 4 implies that x is even the vertex labels of W_4 cannot overlap with those used on P_n.

- Case 2: $n \equiv 1$ (mod 4)

Step 1: Starting with the first vertex of P_n use $1, 3, 5, \ldots, n$ skipping a vertex at each step and wrapping around. This gives the edge labels for P_n : $n+3, n+5, \ldots, 3n-1$ mod $(2n+14)$.

Since these are increasing before reaching the modulus and less than $n+3$ after they exceed the modulus there is no duplication of edge labels.

Step 2: Label the rim vertices of W_4 with $x, x+6, x+10, x+8$ in order and the center of W_4 with $x+4$ where $x = 3n/2 - 3/2$.

This gives the edge labels $2x+4$, $2x+6$, $2x+8$, $2x+10$, $2x+12$, $2x+14$, $2x+16$, $2x+18 = 3n+15$ mod $(2n+14) = n+1$, as desired.

Since $n \equiv 1$ mod 4 implies that x is even the vertex labels of W_4 cannot overlap with those used on P_n. See Fig. 8.

- Case 3: $n \equiv 2$ (mod 4)

Step 1: Label P_n as described in Case 1.

Step 2: Label W_4 as described in Case 1 except $x = 5n/2 + 5$.

This gives the edge labels $2x+4$, $2x+6$, $2x+8$, $2x+10$, $2x+12$, $2x+14$, $2x+16$, $2x+18 = 5n+28$ mod $(2n+14) = n$ as desired.

Since $n \equiv 0$ mod 4 implies that x is even the vertex labels of W_4 cannot overlap with those used on P_n.

- Case 4: $n \equiv 3$ (mod 4)

Step 1: Label the first vertex of P_n with $3n-4$. Starting with the second vertex of P_n use $1, 3, 5, \ldots, 2n-3$ skipping a vertex at each step, wrapping around to the third vertex, and continuing to skip a vertex at each step. See Fig. 9.

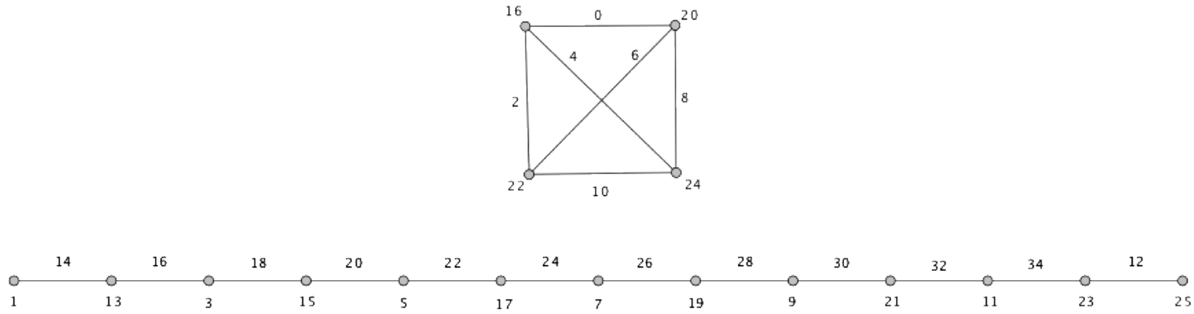

Fig. 10. $K_4 \cup P_{13}$, (mod 36), $n \equiv 1 \pmod 4$, Theorem 3.9.

This gives the edge labels for $P_n :\ 3n-3, n+1, n+3, \ldots, 3n-5 \bmod (2n+14)$. Since these are increasing before reaching the modulus and less than $n+2$ after they exceed the modulus there is no duplication of edge labels.

Step 2: Label the rim vertices of W_4 with $x, x+6, x+10, x+8$ in order and the center of W_4 with $x+4$ where $x = 3n/2 - 5/2$.

This gives the edge labels $2x+4, 2x+6, 2x+8, 2x+10, 2x+12, 2x+14, 2x+16, 2x+18 = 3n+13 \bmod (2n+14) = n-1$ as desired.

Since $n \equiv 3 \pmod 4$, this implies that x is even. Thus the vertex labels of W_4 cannot overlap with those used on P_n. □

Theorem 3.9. *$K_4 \cup P_n$ is properly even harmonious if $n > 1$.*

Proof. The modulus is $2n+10$.

- Case 1: $n \equiv 0 \pmod 4$

  Starting with the first vertex of P_n use $1, 3, 5, \ldots, 2n-1$ skipping a vertex at each step and wrapping around.

  This gives the edge labels of $n+2, n+4, \ldots, 3n-2$ for P_n. Since these are increasing before reaching the modulus and less than $n+2$ after they are equal or exceed the modulus there is no duplication of edge labels.

  Label the vertices of K_4 with $x, x+4, x+8, x+6$ in order, where $x = (3n-4)/2$. This gives the edge labels $2x+4 = 3n, 3n+2, 3n+4, 3n+6, 3n+8, 3n+10$, which is $n \bmod (2n+10)$, as desired.

  Because $n \equiv 0 \pmod 4$ the vertex labels of K_4 are even and therefore cannot overlap with the odd labels of P_n.
- Case 2: $n \equiv 1 \pmod 4$

  Starting with the first vertex of P_n but ignoring the last vertex use $1, 3, 5, \ldots, 2n-3$ skipping a vertex at each step and wrapping around. Label the last vertex of P_n with $n+12$ (see Fig. 10). This gives the edge labels $n+1, \ldots, 3n-5, n-1$ for P_n.

  Label the vertices of K_4 with $x, x+4, x+8, x+6$ in order, where $x = (3n-7)/2$.

  This gives the K_4 edge labels $2x+4 = 3n-3, 3n-1, 3n+1, 3n+3, 3n+5, 3n+7$, which is $n-3 \bmod (2n+10)$, as desired.

  Because $n \equiv 1 \pmod 4$ the vertex labels of K_4 are even and therefore cannot overlap with the odd labels of P_n.
- Case 3: $n \equiv 2 \pmod 4$

  Write $n = 4k+2$. Label P_n as in Case 1 to obtain the edge labels of $P_n : n+2, n+4, \ldots, 3n-2$.

  Label the vertices of K_4 with $x, x+4, x+8, x+6$ in order, where $x = 10k+8$.

  Then $2x+4 = 20k+20 = 5n+10 = 3n \bmod (2n+10)$. So the edge labels of K_4 are: $3n, 3n+2, 3n+4, 3n+6, 3n+8, 3n+10$, which is $n \bmod 2n+10$, as desired.

  Because x is even the vertex labels of K_4 are even and therefore cannot overlap with the odd labels of P_n.
- Case 4: $n \equiv 3 \pmod 4$

  Label the first vertex of P_n with $3n-4$. Starting with the second vertex of P_n use $1, 3, 5, \ldots, 2n-3$ skipping a vertex at each step and wrapping around (see Fig. 11).

  This gives the edge labels of $P_n : 3n-3, n+1, n+3, \ldots, 3n-5 \bmod (2n+10)$.

  Label the vertices of K_4 with $x, x+4, x+8, x+6$ in order, where $x = (3n-5)/2$.

  This gives the edge labels $2x+4 = 3n-1, 3n+1, 3n+3, 3n+5, 3n+7, 3n+9$, which is $n-1 \pmod{2n+10}$, as desired.

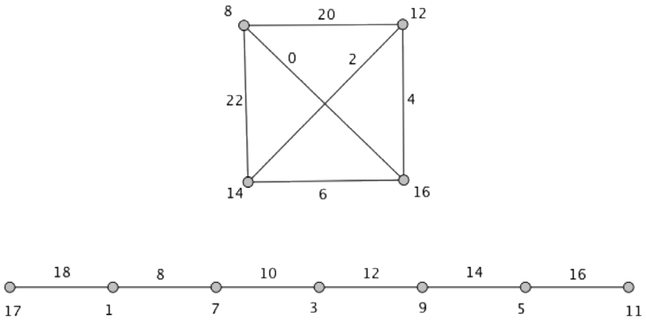

Fig. 11. $K_4 \cup P_7$, (mod 24), $n \equiv 3 \pmod 4$, Theorem 3.9.

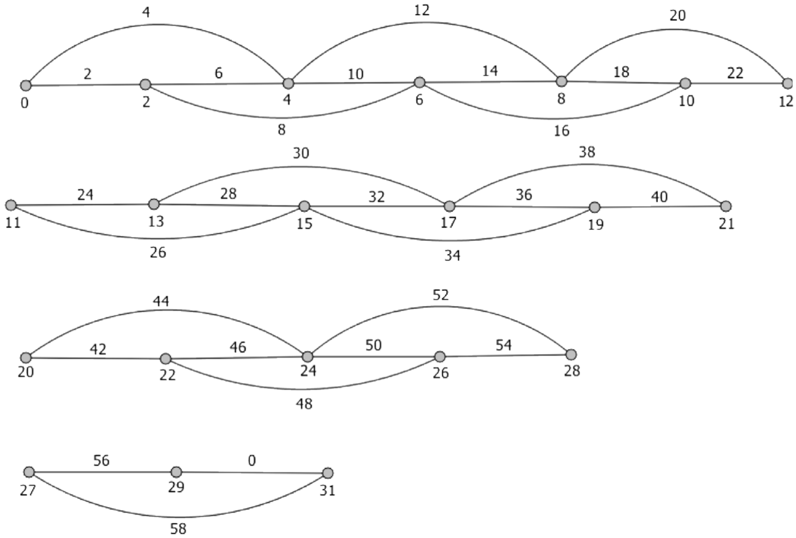

Fig. 12. $P_7^2 \cup P_6^2 \cup P_5^2 \cup P_3^2$, (mod 60), Theorem 3.10.

Because $n \equiv 3 \pmod 4$ the vertex labels of K_4 are even and therefore cannot overlap with the odd labels of P_n. □

Theorem 3.10. *$P_{m_1}^2 \cup P_{m_2}^2 \cup \cdots \cup P_{m_n}^2$ is strongly even harmonious for $m > 2$, $n \geq 1$.*

Proof. The modulus is $4(m_1 + m_2 + \cdots + m_n) - 6n$.

Label the vertices of $P_{m_1}^2$ with $0, 2, \ldots, 2m_1 - 2$.

Label the vertices of $P_{m_2}^2$ with $2m_1 - 3, 2m_1 - 1, \ldots, 2m_1 + 2m_2 - 5$.

Label the vertices of $P_{m_3}^2$ with $2m_1 + 2m_2 - 6, 2m_1 + 2m_2 - 4, \ldots, 2m_1 + 2m_2 + 2m_3 - 8$.

Continue in this fashion and label the vertices of $P_{m_n}^2$ with $2(m_1 + m_2 + \cdots + m_{n-1}) - 3n + 3, 2(m_1 + m_2 + \cdots + m_{n-1}) - 3n + 5, \ldots, 2(m_1 + m_2 + \cdots + m_n) - 3n + 1$ as shown in Fig. 12.

The corresponding edge labels are $2, 4, \ldots, 4(m_1 + m_2 + \cdots + m_n) - 6n = 0$. □

Theorem 3.10 can easily be extended to the union of K_4 or W_4 and the squares of paths. Notice in labeling K_4, we pick up the largest edge labels. This enables us to label $P_{m_1}^2$ such that the first edge label is zero and increasing sequentially from there.

Theorem 3.11. *$K_4 \cup P_{m_1}^2 \cup P_{m_2}^2 \cup \cdots \cup P_{m_n}^2$ is properly even harmonious for $m_i > 2$, $n \geq 1$.*

Proof. The modulus is $4(m_1 + m_2 + \cdots + m_n) - 6n + 12$.

Step 1: Label the vertices of K_4 with $4(m_1 + m_2 + \cdots + m_n) - 6n + 6, 4(m_1 + m_2 + \cdots + m_n) - 6n + 8, 0, 4(m_1 + m_2 + \cdots + m_n) - 6n + 10$. The edge labels are $4(m_1 + m_2 + \cdots + m_n) - 6n, 4(m_1 + m_2 + \cdots + m_n) - 6n + 2, \ldots, 4(m_1 + m_2 + \cdots + m_n) - 6n + 10$.

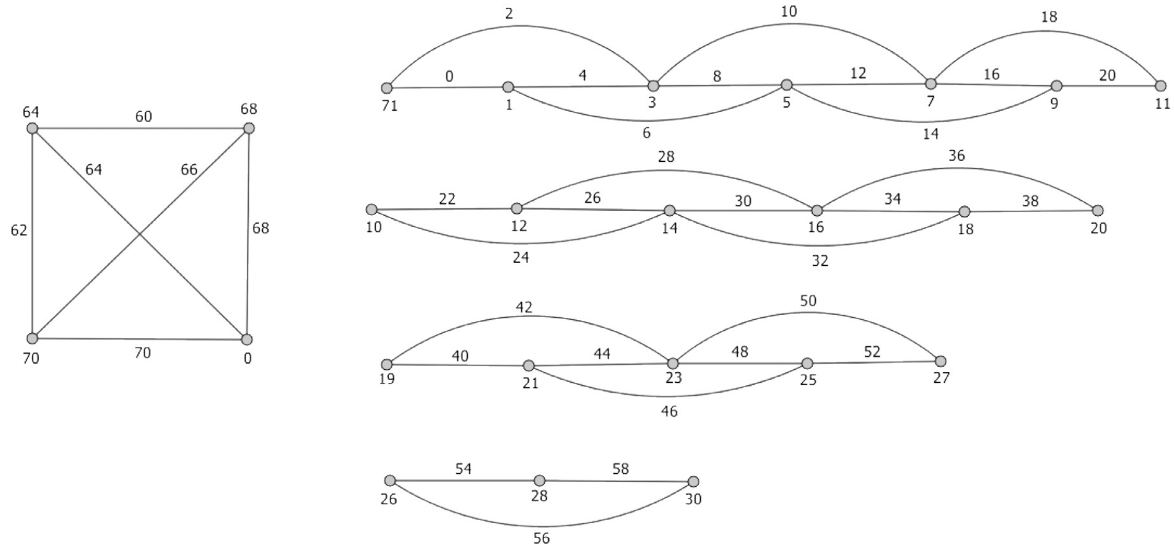

Fig. 13. $K_4 \cup P_7^2 \cup P_6^2 \cup P_5^2 \cup P_3^2$, (mod 72), Theorem 3.11.

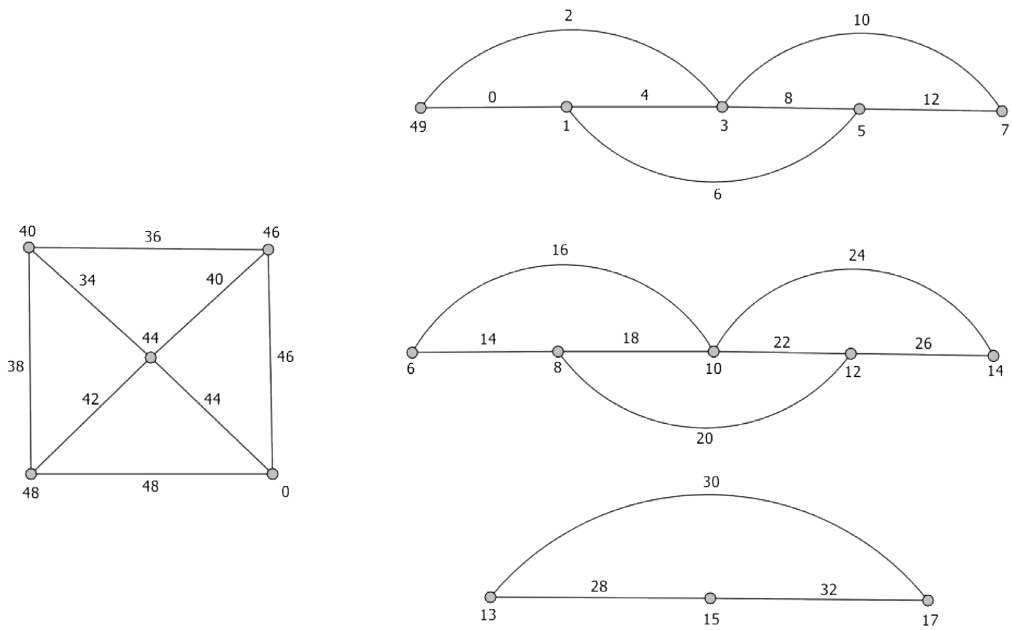

Fig. 14. $W_4 \cup P_5^2 \cup P_5^2 \cup P_3^2$, (mod 50), Theorem 3.12.

Step 2: Label the vertices of $P_{m_1}^2$ with $-1 = 4(m_1 + m_2 + \cdots + m_n) - 6n + 11, 1, 3, \ldots, 2m_1 - 3$.

Label the vertices of $P_{m_2}^2$ with $2m_1 - 4, 2m_1 - 2, \ldots, 2m_1 + 2m_2 - 6$.

Label the vertices of $P_{m_3}^2$ with $2m_1 + 2m_2 - 7, 2m_1 + 2m_2 - 5, \ldots, 2m_1 + 2m_2 + 2m_3 - 9$.

Continue in this fashion and label the vertices of $P_{m_n}^2$ with $2(m_1 + m_2 + \cdots + m_{n-1}) - 3n + 2, 2(m_1 + m_2 + \cdots + m_{n-1}) - 3n + 4, \ldots, 2(m_1 + m_2 + \cdots + m_n) - 3n$. See Fig. 13.

The corresponding edge labels are $0, 2, \ldots, 4(m_1 + m_2 + \cdots + m_n) - 6n - 2$. □

Theorem 3.12. *$W_4 \cup P_{m_1}^2 \cup P_{m_2}^2 \cup \cdots \cup P_{m_n}^2$ is properly even harmonious for $m_i > 2$, $n \geq 1$.*

Proof. The modulus is $4(m_1 + m_2 + \cdots + m_n) - 6n + 16$.

Step 1: Label the interior vertex of W_4 with $4(m_1 + m_2 + \cdots + m_n) - 6n + 10$ and the perimeter vertices with $4(m_1 + m_2 + \cdots + m_n) - 6n + 6, 4(m_1 + m_2 + \cdots + m_n) - 6n + 12, 0, 4(m_1 + m_2 + \cdots + m_n) - 6n + 14$. The corresponding edge labels are the even integers from $4(m_1 + m_2 + \cdots + m_n) - 6n$ to $4(m_1 + m_2 + \cdots + m_n) - 6n + 14$.

Step 2: Label the vertices of $P^2_{m_1}$ with $-1 = 4(m_1 + m_2 + \cdots + m_n) - 6n + 15, 1, 3, \ldots, 2m_1 - 3$.

Label the vertices of $P^2_{m_2}$ with $2m_1 - 4, 2m_1 - 2, \ldots, 2m_1 + 2m_2 - 6$.

Label the vertices of $P^2_{m_3}$ with $2m_1 + 2m_2 - 7, 2m_1 + 2m_2 - 5, \ldots, 2m_1 + 2m_2 + 2m_3 - 9$.

Continue in this fashion and label the vertices of $P^2_{m_n}$ with $2(m_1 + m_2 + \cdots + m_{n-1}) - 3n + 2, 2(m_1 + m_2 + \cdots + m_{n-1}) - 3n + 4, \ldots, 2(m_1 + m_2 + \cdots + m_n) - 3n$. See Fig. 14.

The corresponding edge labels are $0, 2, \ldots, 4(m_1 + m_2 + \cdots + m_n) - 6n - 2$. □

Acknowledgments

This paper is a modified version of a masters degree thesis done by the second author at University of Minnesota Duluth done under the supervision of the first author [9].

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