Mathematics of Cryptography

A Guided Tour

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Talk Plan

1. Group Introduction
2. Inside RSA
3. Case Study
4. Elliptic Curves

The Tour Starts Here

- This talk will give a guided tour of the mathematics underlying cryptography.
- We'll take apart a related set of public key cryptographic algorithms, to see how they work.
- **Disclaimer:** The algorithms are presented in their simplest form—actual systems would implement much more efficient versions.

Diffie-Hellman Key Exchange

The **Diffie-Hellman key exchange** protocol allows two people to use a public channel to set up a shared secret key:

1. Alice and Bob publically agree on a large prime p and an integer x.
2. Alice randomly picks an integer a, and sends Bob $x^a \bmod p$.
3. Bob randomly picks an integer b, and sends Alice $x^b \bmod p$.
4. Alice and Bob both compute $x^{ab} \bmod p$ and use this as a shared secret key.
    - Alice computes $((x^b \bmod p)^a \bmod p) = (x^{ab} \bmod p)$.
    - Bob computes $((x^a \bmod p)^b \bmod p) = (x^{ab} \bmod p)$.

Modular Multiplication Groups

- Multiplication modulo a prime p forms a **group**:
  - There's an **identity** 1 such that $x * 1 = x$.
  - Each element x has an **inverse** x^{-1} such that $x * x^{-1} = 1$.
  - The **operation** $*$ is associative: $x * (y * z) = (x * y) * z$.
- The **order** $|x|$ of x is the smallest n such that $x^n = 1$.
- **Example:** Multiplication modulo 7:

Operation

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Inverse

	$\cdot^{-1}$
1	1
2	4
3	5
4	2
5	3
6	6

Order

	$\lvert \cdot \rvert$
1	1
2	3
3	6
4	3
5	6
6	2

Group Examples

- **Number groups**
    - Addition of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
    - Multiplication of non-zero real numbers.
- **Permutation groups** (group operation is composition)
    - Substitution ciphers.
    - Card shuffles ($|G| = 52!$, $|\text{riffle}| = 7$).
    - Symmetry groups of regular polygons.
    - Rubik's cube.
- **Product groups** $G \times H$
    - $(x_1, y_1) *_{G \times H} (x_2, y_2) = (x_1 *_G x_2, y_1 *_H y_2)$
    - $1_{G \times H} = (1_G, 1_H)$.
    - $(x, y)^{-1} = (x^{-1}, y^{-1})$.

Group Exponentiation

- Given a group G, we can efficiently compute exponentiation x^n using **repeated squaring**:
  1. If $n = 0$ then return the group identity,
  2. else if n is even then return $(x * x)^{n/2}$,
  3. else return $x * (x^{n-1})$.
- Computing x^n using repeated squaring requires $O(\log n)$ group operations.

The Discrete Logarithm Problem

- Given a group G, the **Discrete Logarithm Problem** tests the difficulty of inverting exponentiation:
  - Given $g, h \in G$, find a k such that $g^k = h$.
- The difficulty of this problem depends on the group G.
  - For addition modulo p, the problem can be solved in $O(\log |G|)$ time.
  - For an ideal black-box group G, solving the discrete logarithm problem requires $O(\sqrt{|G|})$ group operations.
- For multiplication modulo p, the problem is hard.
  - But: The best known algorithm can solve it faster than black-box.
  - And: Odlyzko (1991) broke the secure identification option of the Sun Network File System which used a prime of 192 bits.

Group Encryption: ElGamal

The **ElGamal encryption algorithm** can use any instance $g^k = h$ of the Discrete Logarithm Problem.

1. Alice obtains a copy of Bob's public key (g, h).
2. Alice generates a randomly chosen natural number $i \in \{1, \ldots, |G| - 1\}$ and computes $a = g^i$ and $b = h^i m$.
3. Alice sends the encrypted message (a, b) to Bob.
4. Bob receives the encrypted message (a, b). To recover the message m he uses his private key k to compute

$$a^{-k} b = (g^i)^{-k} h^i m = g^{-ik} (g^k)^i m = g^{ki-ik} m = m \ .$$

Subgroups

- A group H is a **subgroup** of a group G if $H \subseteq G$ and H has the same operation, inverse and identity.
  - **Example:** Integer addition is a subgroup of real addition.
  - **Example:** Substitution ciphers mapping $A \mapsto A$ are a subgroup of all substitution ciphers.
  - **Non-example:** Substitution ciphers mapping $A \mapsto B$ are not a subgroup of anything (no identity, not a group).
- A group G has two trivial subgroups:
  - the whole group G; and
  - the subgroup consisting of just the identity.

Lagrange's Theorem

- **Theorem:** If H is a subgroup of a finite group G, then $|H|$ divides $|G|$.
    - **Proof:** Define the equivalence relation $g_1 \sim g_2$ iff there exists $h \in H$ such that $h * g_1 = g_2$.
- **Corollary:** For each element $g \in G$, $|g|$ divides $|G|$.
    - **Proof:** Each group element $g \in G$ generates a subgroup $\{g^n \mid 0 \leq n < |g|\}$.
- **Corollary:** For each element $g \in G$, $g^{|G|}$ is the identity.
    - **Proof:** $g^{|G|} = g^{|g|k} = (g^{|g|})^k = 1^k = 1$.

RSA Encryption

1. Bob chooses two large primes p, q and computes $n = pq$.
2. Bob chooses an integer e and computes d such that

$$ed \bmod (p-1)(q-1) = 1 \ .$$

3. Bob publishes (n, e) as his public key.
4. Alice takes her message m and computes $c = m^e \bmod n$.
5. Alice sends c to Bob.
6. Bob receives c and computes

$$c^d \bmod n = (m^e \bmod n)^d \bmod n = m^{ed} \bmod n = m \ .$$

"The Magic Words are Squeamish Ossifrage"

- **Chinese Remainder Theorem:** Multiplication modulo n is the product group of multiplication modulo p and multiplication modulo q.
- The group of multiplication modulo a prime p consists of elements $\{1, \ldots, p-1\}$, and thus has size $p-1$.
- The group G of multiplication modulo n therefore has size $(p-1)(q-1)$, and so

$$\begin{aligned} m^{ed} \bmod n &= m^{k(p-1)(q-1)+1} \bmod n \\ &= m^{k|G|+1} \bmod n \\ &= (m^{|G|} \bmod n)^k m \bmod n \\ &= 1^k m \bmod n \\ &= m \qquad \square \end{aligned}$$

Blum Integers

- **Fact:** Given a prime p such that $p \bmod 4 = 3$, exactly one of x and $-x$ has square roots. If x has square roots, they can be computed by $\pm(x^{(p+1)/4} \bmod p)$.
- A number n is a **Blum integer** if $n = pq$ with p, q primes equal to 3 modulo 4.
- **Theorem:** If n is a Blum integer and x is a square mod n, then x has four square roots and exactly one of these is itself a square mod n. Call this unique square root the **principal square root**.
- **Theorem:** Computing square roots modulo n is RP-equivalent to factoring n.

Bit Commitment

This protocol allows Alice and Bob to fairly flip a coin over a network.

1. Alice randomly picks a large Blum integer $n = pq$ and an integer x.
2. Alice computes $y = x^2 \bmod n$, and $z = y^2 \bmod n$.
3. Alice sends Bob (n, z).
4. Bob has to guess whether y lies in the range $H = (0, \frac{1}{2}n)$ or the range $T = (\frac{1}{2}n, n)$.
5. Bob randomly picks H or T and sends his guess to Alice.
6. Alice sends Bob (p, q, x).

Zero-Knowledge Proof

- Let Alice have a secret: a Hamilton cycle H in a large graph G.
- The bit commitment protocol can be built upon to allow Alice to prove she knows the secret to Bob, but without revealing it:
  1. Alice randomly permutes all the vertex labels on G to create a new graph G'.
  2. She then makes two commitments: the vertex pairing she used $f : G \to G'$; and the new Hamilton cycle $H' = f(H)$.
  3. She sends G' and these commitments to Bob.
  4. Bob randomly chooses either H' or f, and sends his choice to Alice.
  5. Alice sends Bob the information he needs to reveal his choice.

Elliptic Curve Cryptography

- First proposed in 1985 by Koblitz and Miller.
- Part of the 2005 NSA Suite B set of cryptographic algorithms.
- Certicom the most prominent vendor, but there are many implementations.
- Advantages over standard public key cryptography:
  - Known theoretical attacks much less effective,
  - so requires much shorter keys for the same security,
  - leading to reduced bandwidth and greater efficiency.
- However, there are also disadvantages:
  - The algorithms are more complex, so it's harder to implement them correctly.
  - Patent uncertainty surrounding many implementation techniques.

Elliptic Curves

- An elliptic curve is the set of points (x, y) satisfying an equation of the form

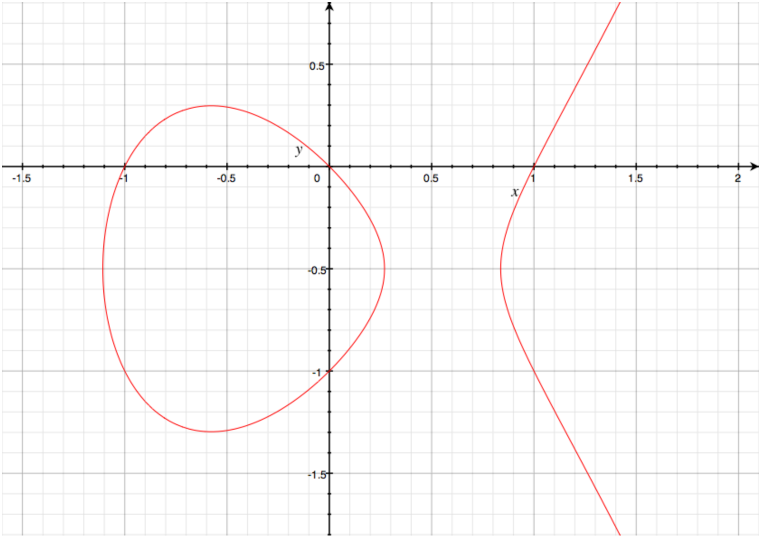
$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \ .$$

- Despite the name, they don't look like ellipses!
- Elliptic curves are used in number theory: Wiles proved Fermat's Last Theorem by showing that the elliptic curve

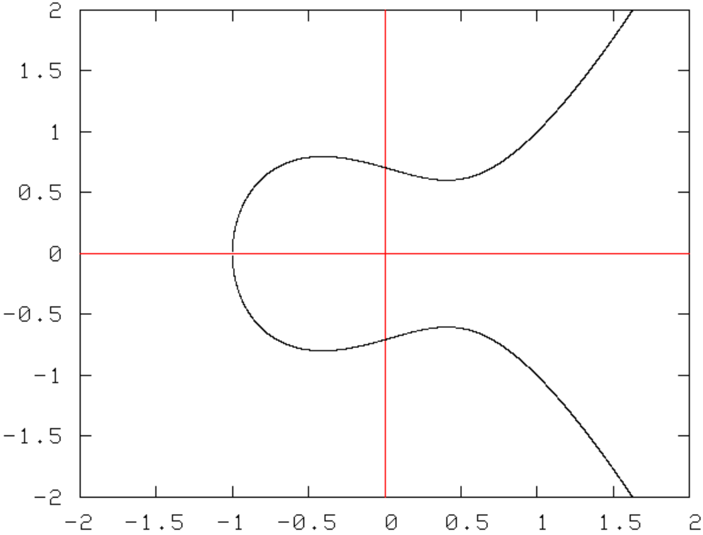
$$y^2 = x(x - a^n)(x + b^n)$$

generated by a counter-example $a^n + b^n = c^n$ cannot exist.

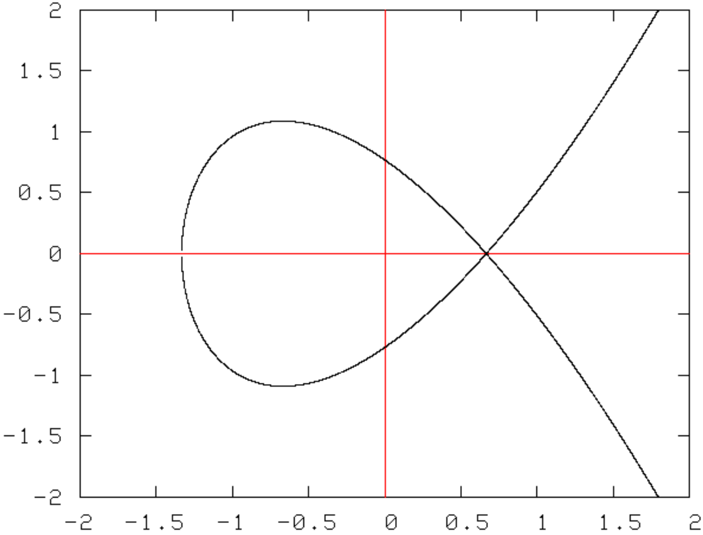
Example Elliptic Curve $y^2 + y = x^3 - x$



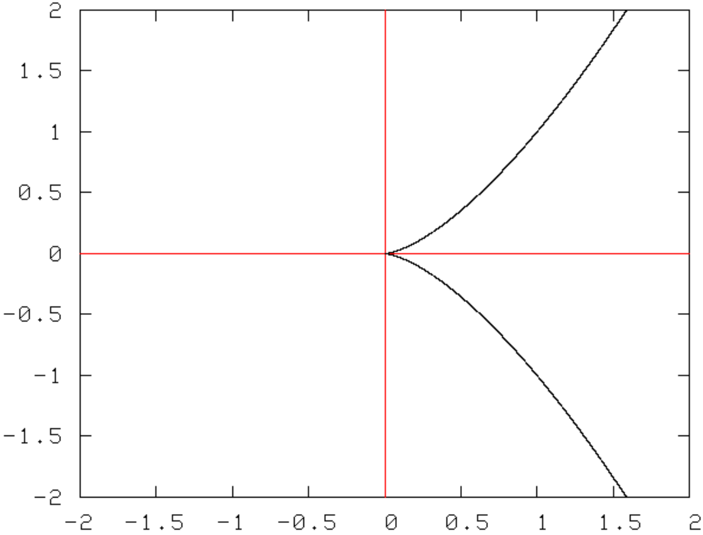
Example Elliptic Curve $y^2 = x^3 - \frac{1}{2}x + \frac{1}{2}$



Example Elliptic Curve $y^2 = x^3 - \frac{4}{3}x + \frac{16}{27}$



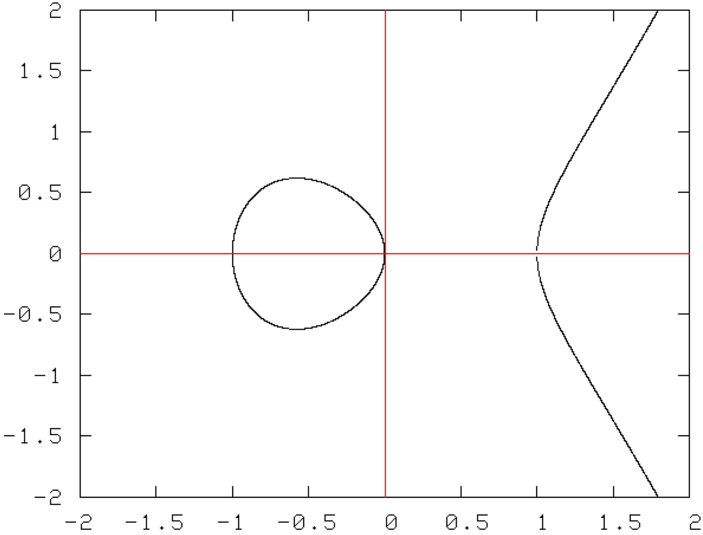
Example Elliptic Curve $y^2 = x^3$



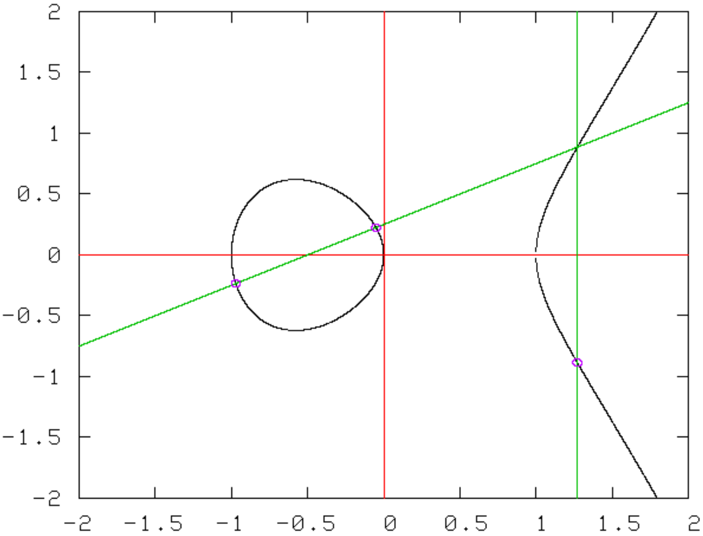
Elliptic Curve Group

- **Fact:** The points (x, y) satisfying the elliptic curve equation form a group.
- It's possible to 'add' two points on an elliptic curve to get a third point on the curve.
- The identity is a special zero point $\mathcal{O}$ *infinitely far up the y-axis*.

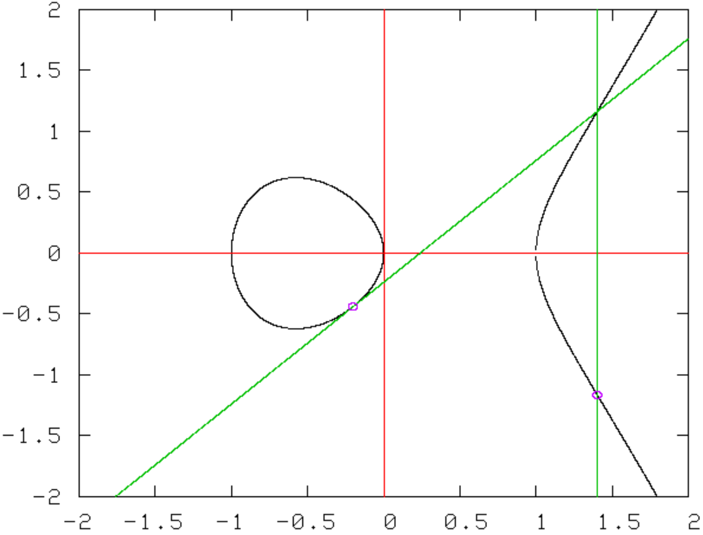
Example Elliptic Curve $y^2 = x^3 - x$



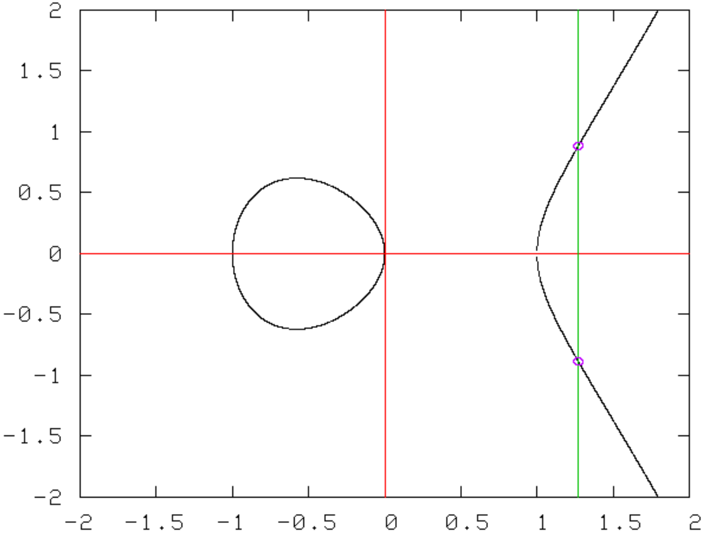
Example Elliptic Curve $y^2 = x^3 - x$: Addition



Example Elliptic Curve $y^2 = x^3 - x$: Doubling



Example Elliptic Curve $y^2 = x^3 - x$: Negation



Elliptic Curve Cryptography

- The graphs showed elliptic curves points (x, y) where x and y were real numbers.
- But the elliptic curve operations can be defined for any underlying field.
- Instead of the geometric definition, use algebra:

$$-(x, y) = (x, -y - a_1 x - a_3) \ .$$

- Elliptic curve cryptography uses finite fields $\mathrm{GF}(p^n)$.
  - $\mathrm{GF}(p)$ is the field $\{0, \ldots, p-1\}$ where all arithmetic is performed modulo the prime p.
  - $\mathrm{GF}(2^n)$ is the field of polynomials where all the coefficients are either 0 or 1.